Prenos varianc in kovarianc – Slepi poligon treh novih točk

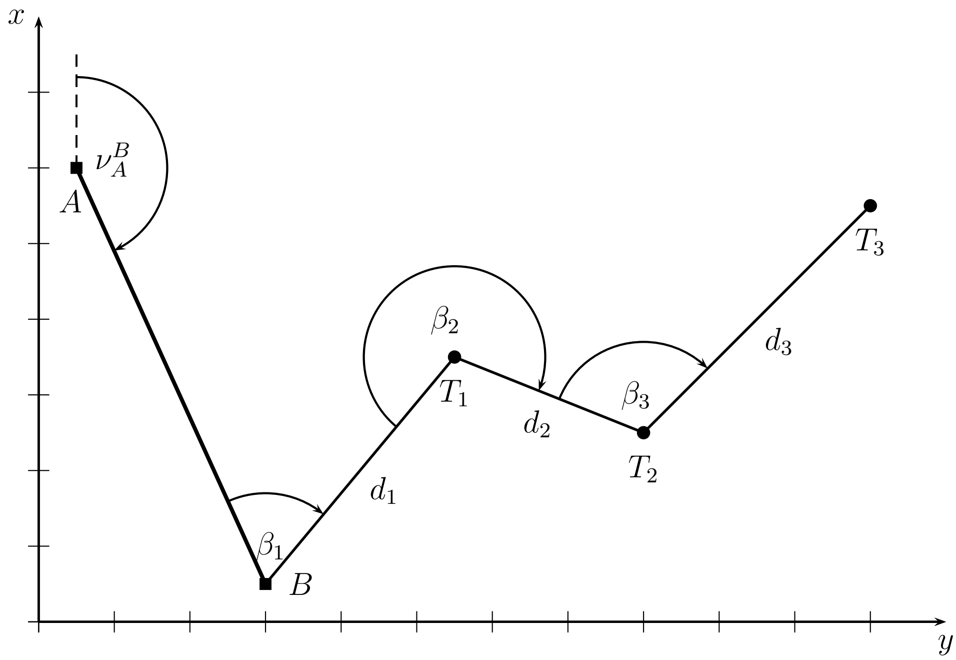
Določiti želimo koordinate treh novih točk (T_1, T_2 in T_3) na osnovi dveh dani točk (A in B) ter opazovanj slepega poligona (d_1, β_1, d_2, β_2, d_3 in β_3), kot to prikazuje slika 1. Dane koordinate točk A in B sta:

- $A(y_A, x_A) = (461\,300.0\,\text{m}, 100\,600.0\,\text{m})$ in
- $B(y_B, x_B) = (461\,400.0\,\text{m}, 100\,550.0\,\text{m})$.

Opazovanja v slepem poligonu so enaka:

- $d_1 = 75.0\,\text{m}, \beta_1 = 100°$,
- $d_2 = 50.0\,\text{m}, \beta_2 = 230°$ in
- $d_3 = 100.0\,\text{m}, \beta_3 = 75°$.

Če je natančnost vseh dolžin enaka $\sigma_d = 5.0\,\text{cm}$ in je enaka tudi natančnost vseh kotov $\sigma_\beta = 2'$, izračunaj koordinate točk $T_1(y_1, x_1)$, $T_2(y_2, x_2)$ in $T_3(y_3, x_3)$. S prenosom varianc in kovarianc izračunaj tudi natančnosti vseh koordinat in vse korelacije med vsemi koordinatami.



Slika 1: Skica slepega poligona dveh danih in treh novih tock

1. Sestavimo vektor opazovanj $\mathbf{x}$ in pripadajočo variančno-kovariančno matriko $\boldsymbol{\Sigma}_{xx}$.
   Število opazovanj je 6, ki jih razvrstimo kot:

$$\mathbf{x} = \begin{bmatrix} d_1 & \beta_1 & d_2 & \beta_2 & d_3 & \beta_3 \end{bmatrix}^{\mathrm{T}} \tag{1}$$

V vektorju $\mathbf{x}$ iz enačbe 1 so dolžinske količine v metrih, kotne količine pa v radianih. Variančno-kovariančna matrika je velikosti 6×6, diagonalna, kjer so na diagonali elementi:

$$\mathbf{\Sigma}_D = \begin{bmatrix} \sigma_d^2 & \sigma_\beta^2 & \sigma_d^2 & \sigma_\beta^2 & \sigma_d^2 & \sigma_\beta^2 \end{bmatrix} \tag{2}$$

Vrednost varianc iz enačbe 2 sta $\sigma_d^2 = 0.0025\,\mathrm{m}^2$ in $\sigma_\beta^2 = 3.385 \times 10^{-7}$.

2. Določimo vse naše neznanke y_j $(j = 1, \ldots, m)$ in sestavimo vektor neznank $\mathbf{y}$.
   Določiti želimo koordinate treh točk novih točk ($T_1(y_1, x_1)$, $T_2(y_2, x_2)$ in $T_3(y_3, x_3)$), torej je $m = 6$ in velja:

$$\mathbf{y} = \begin{bmatrix} y_1 & x_1 & y_2 & x_2 & y_3 & x_3 \end{bmatrix}^\mathrm{T} \tag{3}$$

3. Določimo funkcijske zveze med neznankami in opazovanji, $y_j = f_j(x_1, x_2, x_3, \ldots, x_n)$, $(j = 1, \ldots, m)$ in izračunamo vrednosti neznank $\mathbf{y}$.
   Uporabimo enačbe za izračun točk v slepem poligonu. Velja:

$$\begin{aligned}
y_1 &= y_B + d_1 \sin\left(\nu_B^{T_1}\right) = 461\,444.680\,\mathrm{m} \\
x_1 &= x_B + d_1 \cos\left(\nu_B^{T_1}\right) = 100\,610.239\,\mathrm{m} \\
y_2 &= y_B + d_1 \sin\left(\nu_B^{T_1}\right) + d_2 \sin\left(\nu_{T_1}^{T_2}\right) = 461\,494.590\,\mathrm{m} \\
x_2 &= x_B + d_1 \cos\left(\nu_B^{T_1}\right) + d_2 \cos\left(\nu_{T_1}^{T_2}\right) = 100\,613.234\,\mathrm{m} \\
y_3 &= y_B + d_1 \sin\left(\nu_B^{T_1}\right) + d_2 \sin\left(\nu_{T_1}^{T_2}\right) + d_3 \sin\left(\nu_{T_2}^{T_3}\right) = 461\,462.968\,\mathrm{m} \\
x_3 &= x_B + d_1 \cos\left(\nu_B^{T_1}\right) + d_2 \cos\left(\nu_{T_1}^{T_2}\right) + d_3 \cos\left(\nu_{T_2}^{T_3}\right) = 100\,708.103\,\mathrm{m}
\end{aligned} \tag{4}$$

   V enačbi 4 smerne kote med točkami ($\nu_B^{T_1}$, $\nu_{T_1}^{T_2}$ in $\nu_{T_2}^{T_3}$) izračunamo kot:

$$\begin{aligned}
\nu_B^{T_1} &= \nu_A^B + \beta_1 - 180^\circ = 36^\circ 33' 54'' \\
\nu_{T_1}^{T_2} &= \nu_A^B + \beta_1 + \beta_2 - 2 \cdot 180^\circ = 86^\circ 33' 54'' \\
\nu_{T_2}^{T_3} &= \nu_A^B + \beta_1 + \beta_2 + \beta_3 - 3 \cdot 180^\circ = 341^\circ 33' 54''
\end{aligned} \tag{5}$$

4. Izračunamo vseh $m \times n$ parcialnih odvodov $\frac{\partial f_j}{\partial x_i}$ in sestavimo Jakobijevo matriko $\mathbf{J}$ velikosti $m \times n$.
   Izračunati moramo vse parcialne odvode, vseh 6 neznank moramo odvajati o vseh 6-ih opazovanjih. Imamo torej 36 parcialnih odvodov, ki pa imajo obliko:

   - parcialni odvodi y_1:

$$\begin{aligned}
\frac{\partial y_1}{\partial d_1} &= \sin\left(\nu_B^{T_1}\right) & \frac{\partial y_1}{\partial \beta_1} &= d_1 \cos\left(\nu_B^{T_1}\right) \\
\frac{\partial y_1}{\partial d_2} &= 0 & \frac{\partial y_1}{\partial \beta_2} &= 0 \\
\frac{\partial y_1}{\partial d_3} &= 0 & \frac{\partial y_1}{\partial \beta_3} &= 0
\end{aligned} \tag{6}$$

- parcialni odvodi x_1:

$$\begin{aligned} \frac{\partial x_1}{\partial d_1} &= \cos\left(\nu_B^{T_1}\right) & \frac{\partial x_1}{\partial \beta_1} &= -d_1 \sin\left(\nu_B^{T_1}\right) \\ \frac{\partial x_1}{\partial d_2} &= 0 & \frac{\partial x_1}{\partial \beta_2} &= 0 \\ \frac{\partial x_1}{\partial d_3} &= 0 & \frac{\partial x_1}{\partial \beta_3} &= 0 \end{aligned} \tag{7}$$

- parcialni odvodi y_2:

$$\begin{aligned} \frac{\partial y_2}{\partial d_1} &= \sin\left(\nu_B^{T_1}\right) & \frac{\partial y_2}{\partial \beta_1} &= d_1 \cos\left(\nu_B^{T_1}\right) + d_2 \cos\left(\nu_{T_1}^{T_2}\right) \\ \frac{\partial y_2}{\partial d_2} &= \sin\left(\nu_{T_1}^{T_2}\right) & \frac{\partial y_2}{\partial \beta_2} &= d_2 \cos\left(\nu_{T_1}^{T_2}\right) \\ \frac{\partial y_2}{\partial d_3} &= 0 & \frac{\partial y_2}{\partial \beta_3} &= 0 \end{aligned} \tag{8}$$

- parcialni odvodi x_2:

$$\begin{aligned} \frac{\partial x_2}{\partial d_1} &= \cos\left(\nu_B^{T_1}\right) & \frac{\partial x_2}{\partial \beta_1} &= -d_1 \sin\left(\nu_B^{T_1}\right) - d_2 \sin\left(\nu_{T_1}^{T_2}\right) \\ \frac{\partial x_2}{\partial d_2} &= \cos\left(\nu_{T_1}^{T_2}\right) & \frac{\partial x_2}{\partial \beta_2} &= -d_2 \sin\left(\nu_{T_1}^{T_2}\right) \\ \frac{\partial x_2}{\partial d_3} &= 0 & \frac{\partial x_2}{\partial \beta_3} &= 0 \end{aligned} \tag{9}$$

- parcialni odvodi y_3:

$$\begin{aligned} \frac{\partial y_3}{\partial d_1} &= \sin\left(\nu_B^{T_1}\right) & \frac{\partial y_3}{\partial \beta_1} &= d_1 \cos\left(\nu_B^{T_1}\right) + d_2 \cos\left(\nu_{T_1}^{T_2}\right) + d_3 \cos\left(\nu_{T_2}^{T_3}\right) \\ \frac{\partial y_3}{\partial d_2} &= \sin\left(\nu_{T_1}^{T_2}\right) & \frac{\partial y_3}{\partial \beta_2} &= d_2 \cos\left(\nu_{T_1}^{T_2}\right) + d_3 \cos\left(\nu_{T_2}^{T_3}\right) \\ \frac{\partial y_3}{\partial d_3} &= \sin\left(\nu_{T_2}^{T_3}\right) & \frac{\partial y_2}{\partial \beta_3} &= d_3 \cos\left(\nu_{T_2}^{T_3}\right) \end{aligned} \tag{10}$$

- parcialni odvodi x_3:

$$\begin{aligned} \frac{\partial x_3}{\partial d_1} &= \cos\left(\nu_B^{T_1}\right) & \frac{\partial x_3}{\partial \beta_1} &= -d_1 \sin\left(\nu_B^{T_1}\right) - d_2 \sin\left(\nu_{T_1}^{T_2}\right) - d_3 \sin\left(\nu_{T_2}^{T_3}\right) \\ \frac{\partial x_3}{\partial d_2} &= \cos\left(\nu_{T_1}^{T_2}\right) & \frac{\partial x_3}{\partial \beta_2} &= -d_2 \sin\left(\nu_{T_1}^{T_2}\right) - d_3 \sin\left(\nu_{T_2}^{T_3}\right) \\ \frac{\partial x_3}{\partial d_3} &= \cos\left(\nu_{T_2}^{T_3}\right) & \frac{\partial x_2}{\partial \beta_3} &= -d_3 \sin\left(\nu_{T_2}^{T_3}\right) \end{aligned} \tag{11}$$

Zgornje parcialne odvode vstavimo v jakobijevo matriko **J** in dobimo:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial y_1}{\partial d_1} & \frac{\partial y_1}{\partial \beta_1} & \frac{\partial y_1}{\partial d_2} & \frac{\partial y_1}{\partial \beta_2} & \frac{\partial y_1}{\partial d_1} & \frac{\partial y_1}{\partial \beta_3} \\ \frac{\partial x_1}{\partial d_1} & \frac{\partial x_1}{\partial \beta_1} & \frac{\partial x_1}{\partial d_2} & \frac{\partial x_1}{\partial \beta_2} & \frac{\partial x_1}{\partial d_1} & \frac{\partial x_1}{\partial \beta_3} \\ \frac{\partial y_2}{\partial d_1} & \frac{\partial y_2}{\partial \beta_1} & \frac{\partial y_2}{\partial d_2} & \frac{\partial y_2}{\partial \beta_2} & \frac{\partial y_2}{\partial d_1} & \frac{\partial y_2}{\partial \beta_3} \\ \frac{\partial x_2}{\partial d_1} & \frac{\partial x_2}{\partial \beta_1} & \frac{\partial x_2}{\partial d_2} & \frac{\partial x_2}{\partial \beta_2} & \frac{\partial x_2}{\partial d_1} & \frac{\partial x_2}{\partial \beta_3} \\ \frac{\partial y_3}{\partial d_1} & \frac{\partial y_3}{\partial \beta_1} & \frac{\partial y_3}{\partial d_2} & \frac{\partial y_3}{\partial \beta_2} & \frac{\partial y_3}{\partial d_1} & \frac{\partial y_3}{\partial \beta_3} \\ \frac{\partial x_3}{\partial d_1} & \frac{\partial x_3}{\partial \beta_1} & \frac{\partial x_3}{\partial d_2} & \frac{\partial x_3}{\partial \beta_2} & \frac{\partial x_3}{\partial d_1} & \frac{\partial x_3}{\partial \beta_3} \end{bmatrix} = \\ = \begin{bmatrix} 0.596 & 60.239 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.803 & -44.680 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.596 & 63.234 & 0.998 & 2.996 & 0.000 & 0.000 \\ 0.803 & -94.590 & 0.060 & -49.910 & 0.000 & 0.000 \\ 0.596 & 158.103 & 0.998 & 97.864 & -0.316 & 94.868 \\ 0.803 & -62.968 & 0.060 & -18.287 & 0.949 & 31.623 \end{bmatrix} \tag{12}$$

5. Izračunamo kovariančno matriko neznank $\mathbf{\Sigma}_{yy} = \mathbf{J}\mathbf{\Sigma}_{xx}\mathbf{J}^{\mathrm{T}}$.
   Izračun variančno-kovariančne matrike $\mathbf{\Sigma}_{yy}$ sledi po sestavi variančno-kovariančne matrike $\mathbf{\Sigma}_{xx}$ im jakobijeve matrike $\mathbf{J}$, izpis matrike pa bomo tu, zaradi velikosti matrike, izpustili. Vsi rezultati so podani v nadaljevanju.

6. Iz variančno-kovariančne matrike neznank $\mathbf{\Sigma}_{yy}$ izračunamo natančnosti neznank σ_j $(j = 1, \dots, m)$ in korelacije med neznankami $\rho_{i,j}$ $(i, j = 1, \dots, m \wedge i \neq j)$.
   Izračunana natančnost koordinat za vsako točko in pripadajoča korelacija so:

$$\begin{aligned} \sigma_{y_1} &= 0.046\,\mathrm{m} & \sigma_{x_1} &= 0.048\,\mathrm{m} & \rho_{y_1x_1} &= 0.130 \\ \sigma_{y_2} &= 0.069\,\mathrm{m} & \sigma_{x_2} &= 0.074\,\mathrm{m} & \rho_{y_2x_2} &= -0.143 \\ \sigma_{y_3} &= 0.136\,\mathrm{m} & \sigma_{x_3} &= 0.075\,\mathrm{m} & \rho_{y_3x_3} &= -0.232 \end{aligned} \tag{13}$$

Vse ostale korelacije lahko zapišemo v pregledni obliki:

	y_1	x_1	y_2	x_2	y_3	x_3
y_1	1.00					
x_1	0.13	1.00				
y_2	0.69	0.07	1.00			
x_2	−0.21	0.86	−0.14	1.00		
y_3	0.66	−0.18	0.74	−0.53	1.00	
x_3	−0.03	0.71	−0.00	0.71	−0.23	1.00

Korelacije v zgornji preglednici so podane za vse možne kombinacije neznank. V sivem so korelacije neznanke s samimi seboj, zato vedno vrednost 1. Modre so korelacije med koordinatama iste točke.