Ordnance Survey

# The ellipsoid and the Transverse Mercator projection

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# Shape of the Earth, constants, formulae and methods

## Contents



### Trade marks

# Shape of the Earth, constants, formulae and methods

## I Introduction

This pamphlet supersedes both the HMSO booklet *Constants Formulae and Methods Used in the Transverse Mercator Projection*, which is out of print, and the Ordnance Survey pamphlet *Transverse Mercator Projection*.

Because of the increasing use of satellite navigation systems a short description of the shape of the Earth and its mathematical representation are given as well as the constants for the projection.

## 2 The shape of the Earth

### 2.1 The nature of the Earth

The Earth is a planet orbiting the Sun and spinning on its axis with 72% of its surface covered by the oceans and the remainder covered by the continents. The continental surface varies greatly in texture and profile unlike the oceans where, with sea water being fairly uniform, both texture and profile are approximately constant.

### 2.2 The Earth's mass and gravity

The Earth has mass and a centre of mass. Masses exert a gravitational attraction whose force is related to the distance from the point in question to the centre of mass. Therefore, gravity tends to decrease with distance from the mass. The ocean surfaces deform under many temporal effects such as tides, currents, weather and so on, many of which are known and can be modelled mathematically. The removal of the ocean topography leaves a surface primarily influenced by gravitation; in fact a surface where gravitational potential is constant (an equipotential surface). A definition of an equipotential surface is that in moving from one point on the surface to another no work is done against gravity. An equipotential surface is convex everywhere and can be defined mathematically. Many such surfaces can be drawn, each with its own numerical value of gravitational potential. Although all these surfaces have the Earth's centre of mass as their centre, they are not parallel to each other.

Most measurements on the surface of the Earth are affected by gravity, including the direction of a plumb-line and the horizontal surface defined by a spirit level. The direction of the plumb-line (or vertical) crosses all equipotential surfaces centred at the mass centre of the earth at right angles. Thus, the vertical is generally a gently curving line.

### 2.3 The geoid

We can choose an equipotential surface which is nearest to the average level of the ocean surface and call it the geoid.

Heights referred to sea level are therefore related to the geoid and are termed orthometric heights. Sea level changes with time and so most countries define the height of one point with respect to mean sea level at some time period and refer to this point as the levelling datum.

Positions determined by geodetic astronomy are related to the geoid in as much as the instrument's axes are aligned in the direction of, and at right angles to, the direction of gravity.

### 2.4 The ellipsoid

No explicit geometrical shape of the geoid has been so far mentioned, except for the fact that it is convex. The mathematics of computing the geoid are complicated and various approximations to its shape have been made. As a first attempt, a spherical Earth introduced negligible error for some cartographic purposes. The attraction of this choice is that it is a surface with constant curvature. An ellipse of rotation, with a semi-minor axis in the polar plane and semi-major axis in the equatorial plane, is a better match to the geoid. This shape (the ellipsoid) has been used for geodetic purposes for over two hundred years. A variety of ‘best-fitting’ ellipsoids (mainly national, regional or global), each with a different size and orientation to the Earth’s spin axis, have been used.

### 2.5 Projections

A projection is required to transfer measurements and positions from the ellipsoid onto a flat surface suitable for making into a map. Ordnance Survey uses a modified version of the Transverse Mercator projection.

In the simple Transverse Mercator projection the surface of the ellipsoid chosen to represent the Earth is represented on a cylinder which touches the ellipsoid along a chosen meridian and which is then unwrapped. The scale is therefore correct along this central meridian and increases on either side of it.

The modification to the projection is to make the scale too small on the central meridian by a factor of 0.9996 approximately. The projection then becomes correct in scale on two lines nearly parallel with and on either side of the central meridian and about two thirds of the way between it and the edges of the projection. On the edges, the projection scale will have increased to approximately 1.0004 of nominal figure. The change in scale is most conveniently done in practice by applying the central meridian scale ($F_0$) to all dimensions of the ellipsoid before calculating the projection. This modification effectively doubles the width of the area over which the projection can be usefully applied.

# 3 Symbols and definitions

$a$ = major semi-axis of ellipsoid
$b$ = minor semi-axis of ellipsoid
$e$ = eccentricity

$$e^2 = \frac{a^2 - b^2}{a^2}$$

$$n = \frac{a - b}{a + b}$$

$\nu$ = radius of curvature at latitude $\phi$ perpendicular to a meridian

$$= \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}}$$

$\rho$ = radius of curvature of a meridian at latitude $\phi$

$$= \frac{a(1 - e^2)}{(1 - e^2\sin^2\phi)^{3/2}} = \frac{\nu(1 - e^2)}{(1 - e^2\sin^2\phi)}$$

$$\eta^2 = \frac{\nu}{\rho} - 1$$

$\phi$ = Latitude of a point.
$\lambda$ = Longitude of a point measured east (+) or west (–) of Greenwich.
$H$ = Height of a point above ellipsoid (ellipsoidal height).
$h$ = Height of a point above sea level / geoid (orthometric height).
$\phi'$ = Latitude of the foot of the perpendicular drawn from a point on the projection to the central meridian.
$\phi_o$ = Latitude of true origin.
$\lambda_o$ = Longitude of true origin.
$E_o$ = Grid eastings of true origin.
$N_o$ = Grid northings of true origin.
$E$ = Grid eastings (metres).
$N$ = Grid northings (metres).
$E_t = E - E_o$ 'true' easting.
$N_t = N - N_o$ 'true' northing.
$F_o$ = Scale factor on the central meridian.
$F$ = Scale factor at a point. (This is usually called local scale factor).
$S$ = True distance between two points on the ellipsoid.
$s$ = Straight-line distance between two points on the projection.
$A$ = True meridional arc.
$M$ = Developed meridional arc = $A \times F_o$.
$t$ = Straight-line direction joining two points on the projection.
$T$ = Direction on the projection of the projected geodesic or line of sight joining two points.
$C$ = 'Convergence' of meridians on the projection that is, the angle at any point in the projection between the north-south grid line and the meridian at that point.
$P = \lambda - \lambda_o$.

# 4 Coordinate systems

## 4.1 Concepts

Positions on the Earth are described numerically and unambiguously, making archiving and computation more straight forward. Any point on the Earth's surface can either be referred to the graticule of latitude and longitude (curvilinear coordinates) on the computation surface (the ellipsoid), or a three dimensional cartesian system with an origin at the Earth's centre of mass. Rectangular Cartesian coordinates are easier to manipulate than curvilinear coordinates but give no concept of height above sea level.

Each system has its uses. Three reasons for ellipsoidal coordinates are:

- They are commonly used and accepted in geometrical geodesy.
- They make use of closed formulae, meaning that definition is exact.
- An ellipsoid positioned close to mean sea level is the first approximation to W, the gravity potential.

## 4.2 Ellipsoidal coordinates

There are many ellipsoids on which geodetic coordinates can be expressed. The positioning of the ellipsoid relative to the Earth's surface is as arbitrary as the selection of the ellipsoid itself. The defining parameters of a geodetic reference system require both ellipsoid and datum to be given. These are:

- Length of semi major axis (a).
- Flattening of ellipsoid (f).
- Geodetic latitude and longitude of the origin ($\phi$ $\lambda$).
- Geoidal separation at the origin (N).
- Two parameters which align the minor axis to the spin axis (CIO/BIH).

The last two are fulfilled by the Laplace condition:

$$\text{Geodetic Azimuth} = \text{Astronomic Azimuth} - (\lambda_A - \lambda_G)\sin\phi_G$$

where $\lambda_A$ is the Astronomic longitude, $\lambda_G$ is the Geodetic longitude and $\phi_G$ is the Geodetic latitude.

## 4.3 Cartesian coordinates

Cartesian systems are referred to an assumed Earth centre. Although terrestrial ellipsoidal coordinates are implicitly referred to an inertial frame by the use of the CIO/BIH system, the definition of a geocentric datum is more explicit. The parameters defining a Cartesian system may be:

- Earth's gravitational constant (GM).
- Earth's angular velocity ($\omega$).
- Speed of light (c).
- Coordinate set of defined terrestrial points C{P|x,y,z}.

Recently, ellipsoids have been defined which use elements of both systems. For instance, WGS84 (and also GRS80) has as its defining parameters:

- Semi major axis (*a*).
- Second degree normalised zonal coefficient of the geopotential ($\underline{C}_2$,0).
- Angular velocity of the Earth ($\omega$).
- Earth's gravitational constant (GM).

# 5. Coordinate conversions

## 5.1 Ellipsoidal ➜ Cartesian

The following formulae, quoted in the text books (for example Bomford G; *Geodesy*, 4th Edition OUP 1980), are suitable for conversion of ellipsoidal to Cartesian coordinates. The X-axis is defined as being parallel to the conventional zero meridian of Greenwich, the Z-axis parallel to the CIO and the Y-axis at right angles to these two (eastwards). The Cartesian system may be geocentric or referred to the vertical at some specified point.

$$X = (v+N+h)\cos\phi\cos\lambda \qquad \mathit{5.1}$$
$$Y = (v+N+h)\cos\phi\sin\lambda \qquad \mathit{5.2}$$
$$Z = ((1-e^2)v+N+h)\sin\phi \qquad \mathit{5.3}$$

Notice that the height of the point is (N+*h*) above the ellipsoid. Thus, equations *5.1*, *5.2* and *5.3* presuppose knowledge of the geoid/ellipsoid separation (N).

## 5.2 Cartesian ➜ ellipsoidal

The reverse transformation, from Cartesian to ellipsoidal coordinates, does not produce such closed formulae.

$$\tan\lambda = \frac{Y}{X} \qquad 5.4$$

$$\tan\phi = \frac{Z + e^2\nu\sin\phi}{(X^2 + Y^2)^{½}} \qquad 5.5$$

Approximate values for $\phi$ and $\nu$ are used and iteration gives rapid convergence.

A closed formula due to Bowring is:

$$\tan\phi = \frac{Z + (e')^2 b\sin^3\theta}{p - e^2 a\cos^3\theta} \qquad 5.6$$

Where $(e')^2$, the second eccentricity$^2$ = $\dfrac{e^2}{(1 - e^2)}$

$$p = (X^2 + Y^2)^{½}$$

$$\tan\theta = \frac{Za}{pb}$$

$$H = (N + h) = \frac{p}{\cos\phi} - \nu \qquad 5.7$$

or $(N + h) = (X^2 + Y^2)^{½}\sec\phi - \nu$ using Bomford's notation.

It is important to note that N needs to be known and that $(N + h)$ is determined when using equation *5.7*. This is the ellipsoidal height and is not a height obtained by spirit levelling.

Direct ellipsoid to ellipsoid conversions are possible using differential equations (Heiskanen W A & Moritz H; *Physical Geodesy*, TU Graz, 1979). However, it is simpler to change ellipsoids via Cartesian coordinates.

In summary, to convert from:

- Ellipsoidal ➜ Cartesian : use equations *5.1*, *5.2* and *5.3*
- Cartesian ➜ ellipsoidal : use equations *5.4*, *5.5* and *5.7*

# 6 Useful ellipsoidal constants

| | **National projection** | **UTM** | **OSGRS80® grid** |
|---|---|---|---|
| $a \times F_0$ | 6 375 020.481 | 6 375 836.645 | 6 375 593.856 |
| $b \times F_0$ | 6 353 722.490 | 6 354 369.181 | 6 354 217.697 |
| $n$ | 0.001673220250 | 0.001686340651 | 0.001679220406 |
| $e^2$ | 0.006670539762 | 0.006722670062 | 0.00669438002290 |

# 7 Constants

| | National projection | Universal Transverse Mercator projection (UTM) | | OSGRS80 grid |
|---|---|---|---|---|
| Ellipsoid | Airy | International (1924) | | GRS80 |
| *a* | 6 377 563.396 m | 6 378 388.000 m | | 6 378 137.000 m |
| *b* | 6 356 256.910 m | 6 356 911.946 m | | 6 356 752.3141 m |
| True origin | Lat 49° *N*<br>Long 2° *W* | Zone 30<br>Lat 0°<br>Long 3° W | Zone 31<br>Lat 0°<br>Long 3° E | Lat 49° *N*<br>Long 2° *W* |
| False origin | *E* 400 000 m *W* of true origin<br><br>*N* 100 000 m N of true origin | *E* 500 000 m *W* of true origin<br><br>*N* 0 m | | *E* 400 000 m *W* of true origin<br><br>*N* 100 000 m N of true origin |
| Grid coordinates of true origin<br>$E_o$<br>$N_o$ | 400 000 m<br>-100 000 m | 500 000 m<br>0 m | | 400 000 m<br>–100 000 m |
| Scale on central meridian ($F_o$) | 0.9996012717 | 0.9996 | | 0.9996012717 |

# 8 Formulae

The formulae given below assume that the linear quantities (including *a* and *b*) are already in international metres *and that they have been scaled by* $F_o$. All angles are expressed in radians, including $\phi$ and $\lambda$ . To convert angles to radians multiply by $\pi/180$ that is, a factor of 0.017453293 or 57.29577951 degrees to a radian.

## 8.1 Developed arc of a meridian from $\phi_2$ to $\phi_1$

$$M_{\phi 2} - M_{\phi 1} = b \left\{ \begin{array}{l} \{(1 + n + \frac{5}{4}n^2 + \frac{5}{4}n^3)(\phi_2 - \phi_1)\} \\ - \{(3n + 3n^2 + \frac{21}{8}n^3)\sin(\phi_2 - \phi_1)\cos(\phi_2 + \phi_1)\} \\ + \{(\frac{15}{8}n^2 + \frac{15}{8}n^3)\sin 2(\phi_2 - \phi_1)\cos 2(\phi_2 + \phi_1)\} \\ - \{\frac{35}{24}n^3 \sin 3(\phi_2 - \phi_1)\cos 3(\phi_2 + \phi_1)\} \end{array} \right\}$$

## 8.2 *E* and *N* from $\phi$ and $\lambda$

Calculate *M* from the equation given above in 8.1 by making $\phi_1$ equal to $\phi_o$ and $\phi_2$ equal to the latitude of the point. Remember to scale *b* by $F_o$. Calculate $\nu$, $\rho$, $\eta^2$ and *P* from the equations in paragraph 3 above, remembering to scale *a* and *b* by $F_o$.

$$\mathrm{I} = M + N_o$$

$$\mathrm{II} = \frac{\nu}{2}\sin\phi\cos\phi$$

$$\mathrm{III} = \frac{\nu}{24}\sin\phi\cos^3\phi(5 - \tan^2\phi + 9\eta^2)$$

$$\mathrm{IIIA} = \frac{\nu}{720}\sin\phi\cos^5\phi(61 - 58\tan^2\phi + \tan^4\phi)$$

Then $N = (\mathrm{I}) + P^2(\mathrm{II}) + P^4\,(\mathrm{III}) + P^6\,(\mathrm{IIIA})$

$$\mathrm{IV} = \nu\cos\phi$$

$$\mathrm{V} = \frac{\nu}{6}\cos^3\phi\left(\frac{\nu}{\rho} - \tan^2\phi\right)$$

$$VI = \frac{\nu}{120}\cos^5\phi(5 - 18\tan^2\phi + \tan^4\phi + 14\eta^2 - 58\tan^2\phi\eta^2)$$

Then $E = E_o + P(\mathrm{IV}) + P^3(\mathrm{V}) + P^5(\mathrm{VI})$

## 8.3 $\phi$ and $\lambda$ from $E$ and $N$

The value of $\phi'$ must first be calculated using the iterative process described below.

i) Calculate an initial value for $\phi'$ using:

$$\phi' = \left(\frac{N - N_0}{a}\right) + \phi_0$$

Remember to scale $a$ by $F_0$.

ii) From paragraph 8.1, calculate $M$, using $\phi'$ for $\phi_2$ and $\phi_0$ for $\phi_1$. Remember to scale $b$ by $F_0$.

iii) Calculate a new value for $\phi'$ using:

$$\phi'_{\mathrm{new}} = \left(\frac{N - N_0 - M}{a}\right) + \phi_{\mathrm{old}}$$

Remember to scale a by $F_0$

iv) From paragraph 8.1, recalculate $M$, using $\phi'_{\mathrm{new}}$ for $\phi_2$ and $\phi_o$ for $\phi_1$.

v) If $(N - N_0 - M)$ is zero or close to zero (say <0.001) then use the most current value of $\phi'$ to calculate $\nu$, $\rho$ and $\eta^2$ and then calculate latitude ($\phi$) and longitude ($\lambda$) from equations VII to XIIA below.

vi) If $(N - N_0 - M)$ is not zero or close to zero then go back to step iii of this process and perform the iteration again.

$$VII = \frac{\tan\phi'}{2\rho\nu}$$

$$VIII = \frac{\tan\phi'}{24\rho\nu^3}(5 + 3\tan^2\phi' + \eta^2 - 9\tan^2\phi'\eta^2)$$

$$IX = \frac{\tan\phi'}{720\rho\nu^5}(61 + 90\tan^2\phi' + 45\tan^4\phi')$$

Then $\phi = \phi' - E_t^2(\mathrm{VII}) + E_t^4(\mathrm{VIII}) - E_t^6(\mathrm{IX})$

$$X = \frac{\sec\phi'}{\nu}$$

$$XI = \frac{\sec\phi'}{6\nu^3}\left(\frac{\nu}{\rho} + 2\tan^2\phi'\right)$$

$$XII = \frac{\sec\phi'}{120\nu^5}(5 + 28\tan^2\phi' + 24\tan^4\phi')$$

$$XIIA = \frac{\sec\phi'}{5040\nu^7}(61 + 662\tan^2\phi' + 1320\tan^4\phi' + 720\tan^6\phi')$$

Then $\lambda = \lambda_0 + E_t(\mathrm{X}) - E_t^3(\mathrm{XI}) + E_t^5(\mathrm{XII}) - E_t^7(\mathrm{XIIA})$

## 8.4 $C$ from $\phi$ and $\lambda$

$$XIII = \sin\phi$$

$$XIV = \frac{\sin\phi\cos^2\phi}{3}(1 + 3\eta^2 + 2\eta^4)$$

$$XV = \frac{\sin\phi\cos^4\phi}{15}(2 - \tan^2\phi)$$

Then $C = P(XIII) + P^3(XIV) + P^5(XV)$

## 8.5 *C* from *E* and *N*

The value of $\phi'$ must first be calculated using the iterative process described above in paragraph 8.3 i to vi.

$$XVI = \frac{\tan\phi'}{\nu}$$

$$XVII = \frac{\tan\phi'}{3\nu^3}(1 + \tan^2\phi' - \eta^2 - 2\eta^4)$$

$$XVIII = \frac{\tan\phi'}{15\nu^5}(2 + 5\tan^2\phi' + 3\tan^4\phi')$$

Then $C + E_t(XVI) - E_t^3(XVII) + E_t^5(XVIII)$

## 8.6 *F* from $\phi$ and $\lambda$

$$XIX = \frac{\cos^2\phi}{2}(1 + \eta^2)$$

$$XX = \frac{\cos^4\phi}{24}(5 - 4\tan^2\phi + 14\eta^2 - 28\tan^2\phi\eta^2)$$

Then $F = F_0(1 + P^2(XIX) + P^4(XX))$

## 8.7 *F* from *E* and *N*

First use the iterative process described above in paragraph 8.3 i to vi to obtain a value for $\phi'$ and hence $\eta^2$, $\rho$ and $\nu$.

$$XXI = \frac{1}{2\rho\nu}$$

$$XXII = \frac{1 + 4\eta^2}{24\rho^2\nu^2}$$

Then $F = F_0(1 + E_t^2(XXI) + E_t^4(XXII))$

## 8.8 (*t* - *T*) from *E* and *N*

1 and 2 are the terminals of the line. Use the iterative process described above in paragraph 8.3 i to vi to obtain a value for $\phi'$ and hence $\rho$ and $\nu$. Use the value:

$$N_m = \frac{N_1 + N_2}{2}$$ in place of $N$ in step i of the iterative process.

$$XXIII = \frac{1}{6\rho\nu}$$

Then $(t_1 - T_1) = (2E_{t1} + E_{t2})(N_1 - N_2)(XXIII)$
$(t_2 - T_2) = (2E_{t2} + E_{t1})(N_2 - N_1)(XXIII)$

The answer is in radians. Convert to seconds by multiplying by $\frac{1}{\sin 1''}$

# 9 Worked examples

All the following examples are based on the Airy ellipsoid and the National projection.

## 9.1 Ellipsoidal ➜ Cartesian

**Caister water tower**

| | |
|---|---|
| Latitude | 52° 39' 27.2531"N |
| Longitude | 1° 43' 4.5177"E |
| N | –0.3 m |
| h | 25.0 m |
| $a$ | 6 377 563.396 |
| $b$ | 6 356 256.910 |
| $e^2$ | 6.67053976E – 03 |
| $\nu$ | 6.39105063E + 06 |
| X | 3 874 938.849 m |
| Y | 116 218.623 m |
| Z | 5 047 168.208 m |

## 9.2 Cartesian ➜ ellipsoidal

**Caister water tower**

| | |
|---|---|
| X | 3 874 938.849 m |
| Y | 116 218.623 m |
| Z | 5 047 168.208 m |
| $a$ | 6 377 563.396 |
| $b$ | 6 356 256.910 |
| $e^2$ | 6.67053976E – 03 |
| $\lambda$ | 1° 43' 4.5177" |
| approximate $\phi$ | 52° 30' |
| $\therefore$ approximate $\nu$ | 6.39099381E + 06 |
| After 5 iterations: | |
| $\nu$ | 6.39105063E + 06 |
| $\phi$ | 52° 39' 27.2531" |
| H | 24.7 |
| Latitude | 52° 39' 27.2531"N |
| Longitude | 1° 43' 4.5177"E |
| Height (H) | 24.7 m |

## 9.3 E, N from latitude, longitude

**Caister water tower**

| | |
|---|---|
| Latitude | 52° 39' 27.2531"N |
| Longitude | 1° 43' 4.5177''E |
| $\nu$ | 6.38850233E + 06 |
| $\rho$ | 6.37275644E + 06 |
| $\eta^2$ | 2.47081362E – 03 |
| $M$ | 4.06688296E + 05 |
| $P$ | 6.48899730E – 02 |
| I | 3.06688296E + 05 |
| II | 1.54040791E + 06 |
| III | 1.560688E + 05 |
| IIIA | –2.0671E + 04 |
| IV | 3.87512057E + 06 |
| V | –1.700008E + 05 |
| VI | –1.0134E + 05 |
| Eastings | 651 409.903 |
| Northings | 313 177.270 |

**Framingham**

| | |
|---|---|
| Latitude | 52° 34' 26.8915"N |
| Longitude | 1° 20' 21.1080"E |
| $\nu$ | 6.38847227E + 06 |
| $\rho$ | 6.37266647E + 06 |
| $\eta^2$ | 2.48024893E – 03 |
| $M$ | 3.97408391E + 05 |
| $P$ | 5.82799762E – 02 |
| I | 2.97408391E + 05 |
| II | 1.54162270E + 06 |
| III | 1.572829E + 05 |
| IIIA | –2.0516E + 04 |
| IV | 3.88249423E + 06 |
| V | –1.685014E + 05 |
| VI | –1.0165E + 05 |
| Eastings | 626 238.248 |
| Northings | 302 646.412 |

## 9.4 Latitude, longitude from *E*, *N*

**Caister water tower**

| | |
|---|---|
| Eastings | 651 409.903 |
| Northings | 313 177.271 |
| $\phi'$ | 52° 42' 57.2785"N |
| $\nu$ | 6.38852334E + 06 |
| $\rho$ | 6.37281931E + 06 |
| $\eta^2$ | 2.46422052E – 03 |
| $M$ | 4.13177271E + 05 |
| $E_t$ | 2.51409903E + 05 |
| VII | 1.61305625E – 14 |
| VIII | 3.339555E – 28 |
| IX | 9.4199E – 42 |
| X | 2.58400625E – 07 |
| XI | 4.698597E – 21 |
| XII | 1.6124E – 34 |
| XIIA | 6.6577E – 48 |
| Latitude | 52° 39' 27.2531"N |
| Longitude | 1° 43' 4.5177"E |

**Framingham**

| | |
|---|---|
| Eastings | 626 238.249 |
| Northings | 302 646.415 |
| $\phi'$ | 52° 37' 16.4305" |
| $\nu$ | 6.38848924E + 06 |
| $\rho$ | 6.37271726E + 06 |
| $\eta^2$ | 2.47492225E – 03 |
| $M$ | 4.02646415E + 05 |
| $E_t$ | 2.26238249E + 05 |
| VII | 1.60757208E – 14 |
| VIII | 3.316668E – 28 |
| IX | 9.3107E – 42 |
| X | 2.57842725E – 07 |
| XI | 4.663699E – 21 |
| XII | 1.5922E – 34 |
| XIIA | 6.5408E – 48 |
| Latitude | 52° 34' 26.8916"N |
| Longitude | 1° 20' 21.1081"E |

## 9.5 Convergence

The convergence $C$ at any point in the projection is the angle between the 'North–South' grid line and the direction of the meridian at that point. If the $(t - T)$ correction is neglected, then Grid Bearing + $C$ = True Bearing. (But see para 8.5). $C$ is zero on the grid line $E = E_0$. It is positive to the east and negative to the west of this grid line. Remember that the meridians converge towards the North Pole which is situated on the $E_0$ grid line.

$C$ from latitude, longitude

**Framingham**

| | |
|---|---|
| Latitude | 52° 34' 26.8915"N |
| Longitude | 1° 20' 21.1080"E |
| $\nu$ | 6.38847227E + 06 |
| $\rho$ | 6.37266647E + 06 |
| $\eta^2$ | 2.48024893E – 03 |
| XIII | 7.94140369E – 01 |
| XIV | 9.849793E – 02 |
| XV | 2.1123E – 03 |
| Convergence | 2° 39' 10.4691" |

**Caister water tower**

| | |
|---|---|
| Latitude | 52° 39' 27.2531"N |
| Longitude | 1° 43' 4.5177"E |
| $\nu$ | 6.38850233E + 06 |
| $\rho$ | 6.37275644E + 06 |
| $\eta^2$ | 2.47081362E – 03 |
| XIII | 7.95024300E – 01 |
| XIV | 9.822941E – 02 |
| XV | 2.0244E – 03 |
| Convergence | 2° 57' 26.5561" |

$C$ from E, N

**Framingham**

| | |
|---|---|
| Eastings | 626 238.249 |
| Northings | 302 646.415 |
| $\phi'$ | 52° 37' 16.4305"N |
| $\nu$ | 6.38848924E + 06 |
| $\rho$ | 6.37271726E + 06 |
| $\eta^2$ | 2.47492225E – 03 |
| XVI | 2.04892047E – 07 |
| XVII | 4.536442E – 21 |
| XVIII | 1.588725E – 34 |
| Convergence | 2° 39' 10.4692'' |

**Caister water tower**

| | |
|---|---|
| Eastings | 651 409.903 |
| Northings | 313 177.271 |
| $\phi'$ | 52° 42' 57.2785" |
| $\nu$ | 6.38852334E + 06 |
| $\rho$ | 6.37281931E + 06 |
| $\eta^2$ | 2.46422052E – 03 |
| XVI | 2.05594320E – 07 |
| XVII | 4.571747E – 21 |
| XVIII | 1.608987E – 34 |
| Convergence | 2° 57' 26.5562'' |

## 9.6 Scale factors and true distance from rectangular coordinates

In order to obtain the true distance ($S$) from the grid distance ($s$) derived from grid coordinates; (or alternatively, in order to convert a true distance measured on the ground to a grid distance for plotting on the map or projection) it is necessary to calculate the scale factor and apply it in the correct sense.

The expression connecting these three quantities is

$$s = S \times F \text{ or } S = \frac{s}{F}$$

The scale factor changes from point to point but so slowly that for most purposes it may be taken as constant within any 10 km square and equal to the value at the centre at the square considered. In the worst case the scale factor changes from one side of a 10 km square to the other by about 6 parts in 100 000. So that a value for the middle of the square would not be in error by more than 1/30 000 for any measurement made in that square.

For all practical purposes the scale factor may be taken as depending only on the distance from the central meridian. In the worst case the variation of scale from North to South of the projection along a line of constant easting is less than 1 in 600 000.

The table of scale factors given at the end of this pamphlet may be used for all ordinary work. Where greater accuracy is required the formula given in paragraph 8.7 may be used.

For a long line the factor should be calculated for the mid point of the line. For lines up to 30 km in length the mid point value will give results with an error not exceeding 1 or 2 parts per million.

If still greater accuracy is needed compute a scale factor for both ends and the mid point and use Simpson's Rule, viz:

$$\frac{1}{F} = \frac{1}{6}\left(\frac{1}{F_1} + \frac{4}{F_m} + \frac{1}{F_2}\right)$$

*F* from latitude, longitude

**Caister water tower**

| | |
|---|---|
| Latitude | 52° 39' 27.2531" |
| Longitude | 1° 43' 4.5177''E |
| $\nu$ | 6.38850233E + 06 |
| $\rho$ | 6.37275644E + 06 |
| $\eta^2$ | 2.47081362E – 03 |
| XIX | 1.844226E – 01 |
| XX | – 1.1032E – 02 |
| Local scale | 1.00037732 |

**Framingham**

| | |
|---|---|
| Latitude | 52° 34' 26.8915" |
| Longitude | 1° 20' 21.1080''E |
| $\nu$ | 6.38847227E + 06 |
| $\rho$ | 6.37266647E + 06 |
| $\eta^2$ | 2.48024893E – 03 |
| XIX | 1.851286E – 01 |
| XX | –1.0879E – 02 |
| Local scale | 1.00022970 |

*F* from E, N

**Caister water tower**

| | |
|---|---|
| Eastings | 651 409.903 |
| Northings | 313 177.271 |
| $\phi'$ | 52° 42'57.2785" |
| $\nu$ | 6.38852334E + 06 |
| $\rho$ | 6.37281931E + 06 |
| $\eta^2$ | 2.46422052E – 03 |
| XXI | 1.228112E – 14 |
| XXII | 2.5385E – 29 |
| Local scale | 1.00037732 |

**Framingham**

| | |
|---|---|
| Eastings | 626 238.249 |
| Northings | 302 646.415 |
| $\phi'$ | 52° 37' 16.4305" |
| $\nu$ | 6.38848924E + 06 |
| $\rho$ | 6.37271726E + 06 |
| $\eta^2$ | 2.47492225E – 03 |
| XXI | 1.228138E – 14 |
| XXII | 2.5388E – 29 |
| Local scale | 1.00022969 |

**Mid point Framingham to Caister water tower**

| | |
|---|---|
| Eastings | 638 824.076 |
| Northings | 307 911.843 |
| $\phi'$ | 52° 40' 6.8552" |
| $\nu$ | 6.38850630E + 06 |
| $\rho$ | 6.37276830E + 06 |
| $\eta^2$ | 2.46957015E – 03 |
| XXI | 1.228125E – 14 |
| XXII | 2.5387E – 29 |
| Local scale | 1.00030156 |

## 9.7 The adjustment of directions on the projection or ($t$ – $T$) correction

The '($t$ – $T$) correction' is the difference between the direction in nature and that on the projection.

The straight line joining the two points in nature (which, neglecting refraction, is practically identical with the geodesic on the ellipsoid) will normally be a curve when plotted on the projection. The difference between the initial direction of that curve and the direction of the straight line joining the two points on the projection is the ($t$ – $T$) correction.

In the figure 1 below two plane triangles ABC are shown one on each side of the central meridian. The curved lines represent the geodesics or lines of sight. *The curved geodesics are always concave towards the central meridian*. AX and AY represent the tangents to the curves at A. The angles BAX and CAY represent the ($t$ – $T$) corrections at A to the lines AB and AC respectively. The sign of the correction for any given case is immediately clear from the diagram.

A similar diagram should always be drawn (with the aid of the rule italicised above) to help in applying the correction in the right sense. This method is less liable to lead to mistakes in sign than is a rule of thumb.

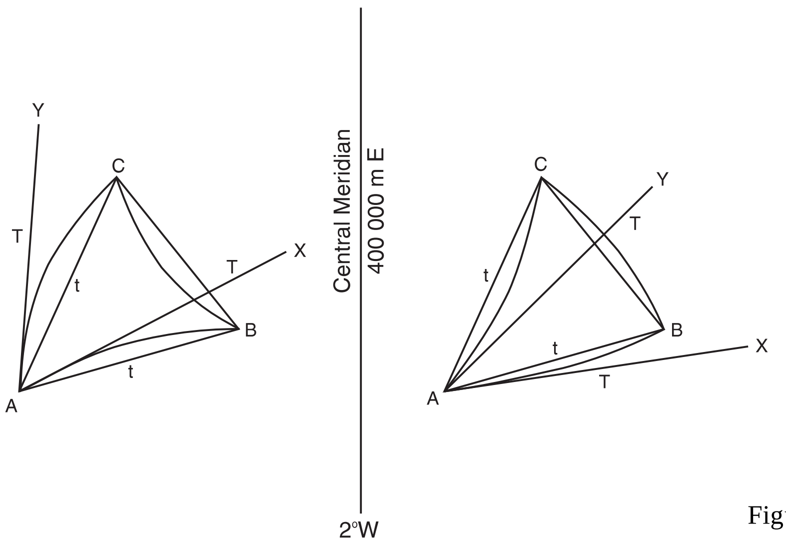


Figure 1

The diagram may lead to confusion in the rare case of a line which crosses the Central Meridian. The ($t$ – $T$) correction is then bound to be very small and would only be considered in first order work of a very precise nature. In such cases a strict algebraic interpretation of the formulae is probably the safest rule.

Since the projection is conformal (that is, directions at a point are maintained relatively correct) it follows that the true bearing of B from A is given by the angle at A between the meridian and the tangent AX in the figure. But the grid bearing of B from A is the angle between grid North and the line AB.

Therefore, if the ($t$  $T$) correction be taken into consideration,

True bearing A ➜ B = grid bearing A ➜ B + C – ($t$ – $T$).

($t$ – $T$) from E, N

**Framingham to Caister water tower**

| | |
|---|---|
| Eastings | 626 238.249 |
| Northings | 302 646.415 |
| Eastings | 651 409.903 |
| Northings | 313 177.271 |
| $N_m$ | 307 911.843 |
| $M$ | 4.07911843E + 05 |
| $\phi'$ | 52° 40' 6.8552" |
| $\nu$ | 6.38850630E + 06 |
| $\rho$ | 6.37276830E + 06 |
| XXIII | 4.09374978E – 15 |
| ($t$ – $T$)a (rads) | –3.0345E – 05 |
| ($t$ – $T$)b (rads) | 3.1430E – 05 |
| | |
| ($t$ – $T$)a (sec) | –6.26 |
| ($t$ – $T$)b (sec) | 6.48 |

## 9.8 True azimuth from rectangular coordinates

The true azimuth is obtained by computing the grid bearing and applying the convergence and the ($t$ – $T$) correction. (Note that for short lines not exceeding 10 km in length the ($t$ – $T$) correction cannot exceed 7" in the worst case on the limit of the projection. For minor surveys therefore it may be neglected).

The drawing of a rough diagram as shown below in figure 2, from which the signs of $C$ and $(t - T)$ can be seen by inspection is strongly recommended.

| | | | |
|---|---|---|---|
| Station 1 | Framingham | *E* 626 238.249 | *N* 302 646.415 |
| Station 2 | Caister water tower | *E* 651 409.903 | *N* 313 177.271 |
| | | $E_2 - E_1$ + 25 171.654 | $N_2 - N_1$ + 10 530.856 |

$$\frac{E_2 - E_1}{N_2 - N_1} = \text{tangent of plane grid bearing 1 to 2} = +2.39027616$$

Plane grid bearing $(\alpha_1) = 67^\circ\ 17'\ 50.759''$

$\sin \alpha_1 = 0.92252080$
$\cos \alpha_1 = 0.38594738$

$$\text{Plane grid distance} = \frac{E_2 - E_1}{\sin \alpha_1} = 27\ 285.730 = \frac{E_2 - E_1}{\cos \alpha_1} = 27\ 285.730 \text{ (Check)}$$

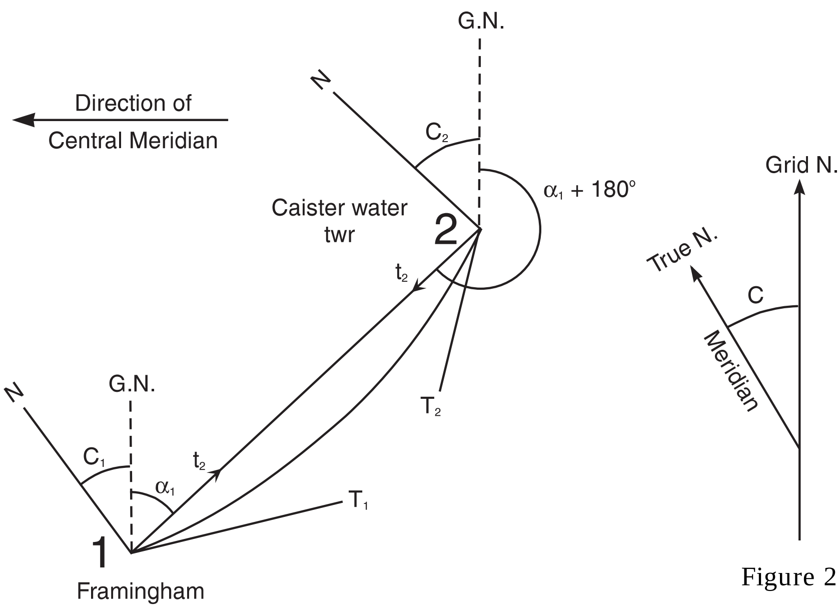


Figure 2

True azimuth$_1$ = $\alpha_1 + C_1 - (t - T)''_1$

True azimuth$_2$ = $\alpha_1 + C_2 - (t - T)''_2 + 180^\circ$

| True azimuth$_1$ | | | True azimuth$_2$ | |
|---|---|---|---|---|
| $\alpha_1$ | $67^\circ\ 17'\ 50''.759$ | from previous examples | $\alpha_1$ | $67^\circ\ 17'\ 50''.759$ |
| $+ C_1$ | $+ 2^\circ\ 39'\ 10''.469$ | | $+ C_2$ | $+ 2^\circ\ 57'\ 26''.556$ |
| $- (t - T)''_1$ | $+ 06''.259$ | | $- (t - T)''_2$ | $- 06''.483$ |
| | | | | $+ 180^\circ\ 00'\ 00''.000$ |
| True azimuth$_1$ = | $69^\circ\ 57'\ 07''.487$ | | True azimuth$_2$ = | $250^\circ\ 15'\ 10''.832$ |

For signs of $C$ and $(t - T)$ see figure 2 above.

## Table of local scale factors

(see paragraph 9.6)

| National Grid easting (km) | | Scale factor F |
|---|---|---|
| 400 | 400 | 0.99960 |
| 390 | 410 | 60 |
| 380 | 420 | 61 |
| 370 | 430 | 61 |
| 360 | 440 | 62 |
| 350 | 450 | 63 |
| 340 | 460 | 65 |
| 330 | 470 | 66 |
| 320 | 480 | 68 |
| 310 | 490 | 70 |
| 300 | 500 | 72 |
| 290 | 510 | 75 |
| 280 | 520 | 78 |
| 270 | 530 | 81 |
| 260 | 540 | 84 |
| 250 | 550 | 88 |
| 240 | 560 | 92 |
| 230 | 570 | 0.99996 |
| 220 | 580 | 1.00000 |
| 210 | 590 | 04 |
| 200 | 600 | 09 |
| 190 | 610 | 14 |
| 180 | 620 | 20 |
| 170 | 630 | 25 |
| 160 | 640 | 31 |
| 150 | 650 | 37 |
| 140 | 600 | 43 |
| 130 | 670 | 1.00050 |

Use of scale factor

$s = S \times F$

Where $s$ = distance in the projection
$S$ = distance on the spheroid at mean sea level
$F$ = Local scale factor from table.


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