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Reference Systems in Satellite Geodesy

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München 2001

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1. Introduction

This summer school deals with satellite navigation systems and their use in science and application. Navigation is concerned with the guidance of vehicles along a chosen path from A to B. Precondition to any navigation is knowledge of position and change of position as a function of time. Thus, navigation requires position determination in real time. It has to combine time keeping and fast positioning. We completely exclude here inertial navigation, i.e. the determination of position changes - while moving - from sensors such as odometers, accelerometers and gyroscopes operating inside a vehicle. The motion of a body has to be determined relative to some reference objects. With inertial navigation methods excluded here, positioning requires direct visibility of these reference objects. Typical measurement elements are ranges, range rates, angles, directions or changes in direction. In some local applications terrestrial markers may serve as reference objects. More versatile reference objects in the past, because of their general visibility, were sun, moon and stars and are artificial satellites today. In principle, positions as a function of time can be deduced directly from the measured elements and relative to the reference objects without any use of a coordinate system. Coordinate systems are not an intrinsic part of positioning and navigation. They are introduced into positioning and navigation as a matter of convenience, elegance and for the purpose of creating order. To a large extent their choice is arbitrary and, again, in many ways a matter of convenience. The description of objects or events in space and time in a coordinate system requires four coordinates, three identifiing the position in space, the fourth providing time. In Newtonian mechanics the time coordinate is independent from the three space coordinates and "absolute". This is not the case when applying special and general relativity. Although part of the coordinate definition in space geodesy is done in the framework of the theory of relativity it is considered beyond the scope of this lecture.

In the course of the centuries, the following hierarchy of three levels of coordinate systems - or more generally - reference systems turned out to be particularily meaningful:

- Space-fixed or inertial systems, in which the positions of stars are fixed or almost fixed and in which the motion of artificial satellites can be formulated according to the Newtonian laws of mechanics.
- Earth-fixed systems, in which all terrestrial points can be expressed conveniently as well as vehicles in motion on the earth's surface.
- Local horizon systems, fixed to observatories or instruments and often oriented horizontally with one axis pointing towards north.

These three levels of reference systems are complemented, when needed, by some specialized ones such as orbit or spacecraft systems or regional terrestrial systems.

Equally important, also time keeping requires some generally adopted reference for maintenance, comparison and transfer of time.

2. Hierarchy of Reference Systems

When talking about reference systems it is useful to distinguish between the three concepts coordinate system, reference system and reference frame.

Coordinate systems are the central - mathematical - element of any geodetic reference system. The choice of a coordinate system in three dimensions requires the definition of its origin (three elements), the orientation of the axes (three elements), and the scale. It is convenient and common practice to choose orthonormal base vectors and the same scale along all three axes. It is again a matter of convenience to tie a system of curvi-linear coordinates, such as spherical, geographical or ellipsoidal coordinates to any such orthonormal system of base vectors. The transformation between coordinates given in two systems consists of a shift of origin (three degrees of freedom) and rotations of the base vectors (three degrees of freedom). We leave aside here the comparison of scale. By definition, time is dealt here as completely independent.

A **reference system** consists of the adopted coordinate system and, in addition, of a set of constants, models and parameters, that are required in order to achieve a certain degree of generality or idealisation. This additional set could be necessary, for example, in order to deal with tectonic plate motion, tides and the earth's response to tides, it could define the constants of a reference ellipsoid or the parameters of a reference gravity field. Since positioning and navigation are global activities, nowadays, it is important that the same set definitions is used everywhere. Thus, internationally adopted conventions are necessary. The International Earth Rotation Service (IERS), a joint service of the International Association of Geodesy (IAG) and of the International Astronomical Union (IAU) prepares the definition of so-called **conventional reference systems** and of their implemention, see e.g. (MCCARTHY, 1996). The conventional international celestial reference system (ICRS) is adopted by IAG and IAU, the conventional terrestrial reference system (CTRS) by IAG done.

Finally, a **reference frame** contains all elements required for the materialization of a reference system in real world. In the case of space fixed or celestial frames it is essentially an adopted catalogue of celestial objects such as stars or quasars, in the case of a terrestrial frame it is the catalogue of coordinates of terrestrial points (stations, observatories) as well as of their velocities. The catalogues are chosen to be consistent with the conventions of the corresponding reference system.

As already said, it makes sense to introduce a hierarchy of three levels of reference systems: space fixed systems, earth fixed systems and local or horizontal systems. They are discussed now in more detail.

2.1 Space-fixed reference systems (or celestial reference systems (CRS)):

They are an approximation of an inertial system. Their purpose is twofold. Celestial objects such as stars or quasars take a fixed position on the celestial sphere. After correction for proper motion their direction can be expressed by two (fixed) coordinates (kinematic part of a CRS). The motion of satellites and of the sun, moon and planets can be described according to the laws of Newtonian mechanics (no apparent forces). This may be denoted the dynamic component of the definition of a CRS.

Definition of the coordinate system:
As origin either the barycentre of the solar system is chosen or the mass centre of the earth. For our further discussion we only consider the geocentric definition. This choice implies that the coordinate triad is accelerated. Its acceleration is mainly caused by the gravitational attraction of sun and moon on the earth. The system is not truely inertial. The base vectors are denoted here $\mathbf{e}_{i=1}$, $i = 1,2,3$ ("*i* " for inertial) with $\mathbf{e}_{i=1}$ and $\mathbf{e}_{i=2}$ defining the {*x, y*}-plane and $\mathbf{e}_{i=3}$ the *z*-axis. The {*x, y*}-plane of a CRS could either be chosen to coincide with the plane of the ecliptic, i.e. the orbit plane of the earth about the sun, or with the equator plane of the earth. Here the latter definition is adopted. The *x*-axis, i.e. base vector of $\mathbf{e}_{i=1}$, points in the direction of the vernal equinox. It is the line of intersection between ecliptic and equator plane. The *z*-axis, base vector $\mathbf{e}_{i=3}$, points into the direction of the mean rotation axis of the earth. The *y*-axis completes a right-handed system. In $\mathbf{e}_i$ the direction to any object/event is expressed by the two angles right ascension α (angle in the equator plane counted from $\mathbf{e}_{i=1}$) and declination δ (elevation angle counted from the equator plane). Due to the torque excerted by sun, moon and planets on the oblate earth and with its equator not in coincidence with the ecliptic plane, the rotating earth undergoes a complicated gyroscopic motion in space. Its steady part is a precession, i.e. a constant rotation of the vernal equinox in the ecliptic plane, with a period of approximately 25800 years or 50."3 per year. Superimposed to this steady precession are shorter period nutations. They are periodic changes of the inclination angle of 23° 26' of the equator plane with respect to the ecliptic. The main period is thereby 18.6 years with an amplitude of 9."2.

As a consequence of the slow rotational motion of the adopted triad $\mathbf{e}_i$ in space, one has to distinguish the apparent or true system, i.e. the instantaneous orientation at a chosen epoch, from a mean system, which is corrected for the periodic contribution of the nutation. Finally, the definition of a conventional reference system, in which celestial objects can be catalogued, requires the definition of a reference epoch. Currently the adopted reference epoch is J2000.0, which is 12^h January 1, 2000 Greenwich time.
In the past the realization of a CRS has been conducted solely by astronomical methods. In recent years very long baseline interferometry (VLBI) and the ESA-satellite HIPPARCOS provided a completely new set of catalogue information, unprecedented in terms of the number of objects, their position accuracy and internal consistency. Some basic information about the realization of the international celestial reference frame (ICRF) is summarized in table 2.1.

Table 2.1: International Celestial Reference System (ICRS) and Frame (ICRF)

(according to IERS conventions)

origin: barycentre of solar system
geocentric origin derived from planetary motion, lunar motion and from artificial satellites

orientation:	in coincidence with ICRF at 1991.25 parallel to axes of FK5 ($\pm$ 8mas) mean equator at J2000.0 *x*-axis: mean vernal equinox at J2000.0
time:	barycentric dynamical time (TDB)
directions:	$\{\alpha, \delta\}$
techniques:	astronomy (FK5) VLBI HIPPARCOS satellite mission
realization:	212 defining quasars (VLBI) 608 quasar positions in total 144 000 star positions of HIPPARCOS catalogue
name:	International Celestial Reference System (ICRS) and Frame (ICRF)
reference:	(McCarthy, 1996)

2.2 Earth-fixed reference systems (or terrestrial reference systems (TRS)):

They serve the description of the position of points on the earth's surface or, in the case of navigation, that of the motion of a vehicle on the earth's surface or close to it. Also geophysical processes such as weather, temperature, magnetic or gravity field are expressed in earth fixed systems. Finally, all our maps are based upon an earth fixed reference system. During the past twenty years, due to the advance of space techniques, precisions in positioning and navigation became so incredibly high that the earth's surface cannot be considered anymore solid and fixed. Instead temporal changes due to surface motions such as tectonic plate motions and deformations due to tides or ocean and atmosphere loading have to be taken into account. This complicates the definition and realisation of an earth fixed reference system severly. On the other hand it implies that such a system can provide a framework for global geophysical monitoning and consequently play a prominent role in earth system studies.

Definition of the coordinate system:
The origin of the coordinate system is the geocentre. The geocentre is thereby defined as the centre of mass of the earth including oceans and atmosphere. The base vectors are denoted $\mathbf{e}_e$, $e = 1,2,3$ ("*e*" for earth-fixed). Again, the base vectors $\mathbf{e}_{e=1}$ and $\mathbf{e}_{e=2}$ define the $\{x, y\}$-plane, $\mathbf{e}_{e=3}$ defines the *z*-axis. The $\{x, y\}$-plane coincides with a conventional equatorial plane of the earth. The base vector $\mathbf{e}_{e=1}$ lies by definition in the Greenwich meridian plane. The base vector $\mathbf{e}_{e=3}$ corresponds to the mean position during 1900 to 1905 of the rotation axis of the earth. This terrestrial pole is denoted conventional terrestrial pole (CTP) or IERS reference pole (IRP). Finally, $\mathbf{e}_{e=2}$ completes the right-handed system. Its evolution in time will not change its orientation relative to the crust, i.e. it will have no residual global rotation with respect to the crust, (McCarthy, 1996).

The coordinate triad $\mathbf{e}_e$ is accompanied by a mean earth ellipsoid with

semi-major axis a = 6 378 137.0 m and
flattening f = 1 / 298.257 222 101.

It allows an easy conversion of cartesian into geographical coordinates.

The realization of the International Terrestrial Reference System (ITRS) is denoted International Terrestrial Reference Frame (ITRF). It consists of an adopted global set of cartesian station coordinates and velocities. Almost annually a new ITRF is produced, based upon newest observations and identified as ITRF.yy, where the numbers (yy) following the designation ITRF specify the last year whose data were used for the realization of the frame. Currently the 2000 version, i.e. ITRF.00 is in preparation. Transformation parameters between the annual realizations are published in the IERS conventions, (McCarthy, 1996). Station coordinates at an arbitrary epoch t are derived from

$$\mathbf{x}(t) = \mathbf{x}(t_o) + \mathbf{v} \cdot (t - t_o) + \sum_i \Delta\mathbf{x}_i(t)$$

with t_o the reference epoch, $\mathbf{v}$ the station velocities and $\Delta\mathbf{x}_i$ coordinate corrections due to various time variable effects, such as those listed in table 2.2.

Table 2.2: List of time variable effects, part of ITRS

- tectonic plate motions (angular velocities of 16 plates, model NNR-NUVEL 1, no-net rotation)
- tides of the solid earth
- loading effects due to ocean loading
- atmospheric loading
- rotational deformation due to polar motion
- postglacial rebound
- instrument effects (antenna deformation, motion of antenna phase centres etc.)

There are several geodetic space techniques from which realizations of the earth fixed reference frame are derived by so-called analysis centres. In a second step and after a careful analysis of all individual solutions, one unique solution is computed taking into account the results of the various centres as well as the various techniques. Such combinations are based on the full variance-covariance error matrices of the individual solutions.

Space techniques that are contributing to ITRS:

- very long baseline interferometry (VLBI)
  (high precision and long term stability)

- satellite laser ranging (SLR)
  (long term stability and geo-centricity)

- lunar laser ranging (LLR)
  (geo-centricity, long term stability, relativistic effects)

- the French tracking system DORIS
  (excellent global station distribution)

- Global Positioning System
  (densest global network, short term stability, high precision).

We see that the various techniques are complementary to each other. Still, optimal combination of all stations and all techniques remains a major challenge.
A summary of ITRS and ITRF is given in table 2.3. It should be added that there exists a further earth fixed reference system: the World Geodetic System 1984 **(WGS84)**. It is a system developed by the U.S. defence mapping agency, specifically for the operational use of the GPS; ITRS can be seen as a refinement of WGS84.

Table 2.3: International Terrestrial Reference System (ITRS) and Frame (ITRF)

(according to IERS conventions)

origin:	mass centre of the earth (including oceans and atmosphere)
scale:	metre
orientation:	in coincidence with BIH-System 1984 ($\pm$ 3 mas)
evolution:	no net rotation (NNR) with respect to crust
ellipsoid:	Geodetic Reference System 1980 (GRS80) a = 6 378 137.0 m f = 1 / 298.257 222 101
directions:	$\{\Phi, \Lambda\}$, $\{\varphi, \lambda\}$, $\{B, L\}$
techniques:	VLBI SLR LLR GPS DORIS
realization:	coordinates $\mathbf{x}(t_o)$ and velocities $\mathbf{v}(t_o)$ at epoch t_o of a large number of instrument locations of geodetic observatories equipped with one up to six techniques in parallel
notation:	ITRF.yy, z. B. ITRF.94
reference:	(McCarthy, 1996)

2.3 Local horizontal reference systems (or topocentric systems): This class of systems is associated with an instrument such as a GPS receiver, a VLBI telescope or a camera. They are therefore topocentric, located in the origin or reference point of the instrument and it is purpose of space positioning to determine the coordinates of this reference point, either in $\mathbf{e}_e$ or in $\mathbf{e}_i$. Local horizontal systems are introduced in order to express the fixed or time variable pointing direction of the instrument to a target point and in order to predict when and under what angles a target will rise or fall. The orientation of base vectors of the local system can either be defined by the local (level) horizontal plane, north direction and plumb line direction (zenith) or, in ellipsoidal or spherical approximation, by the corresponding ellipsoidal or spherical quantities.

Definition of the coordinate system:
The origin of the coordinate system is the instrument origin (topocentre). The base vectors are denoted $\mathbf{e}_l$, $l = 1,2,3$ ("l" for local). Base vectors $\mathbf{e}_{l=1}$ and $\mathbf{e}_{l=2}$ define the {x, y}-plane, $\mathbf{e}_{l=3}$ defines the z-axis. The {x, y}-plane coincides with the local horizon (level surface). It is often approximated by an ellipsoidal or spherical reference surface. The base vector $\mathbf{e}_{l=1}$ points towards north. The z-axis points towards the zenith (or normal of the ellipsoid or sphere). Base vector $\mathbf{e}_{l=2}$ completes the **left-handed** orthonormal triad; it points towards east.

The angle to an object in the horizontal plane counted from north (towards east) is called **azimuth *A***, the angle to an object from the zenith is denoted **zenith distance *z***, its complement to π, the elevation angle above the horizontal plane is called **elevation angle *b***. Thus, when tracking a satellite at a station its changing direction in the horizontal system is expressed by the angles {A, z}.

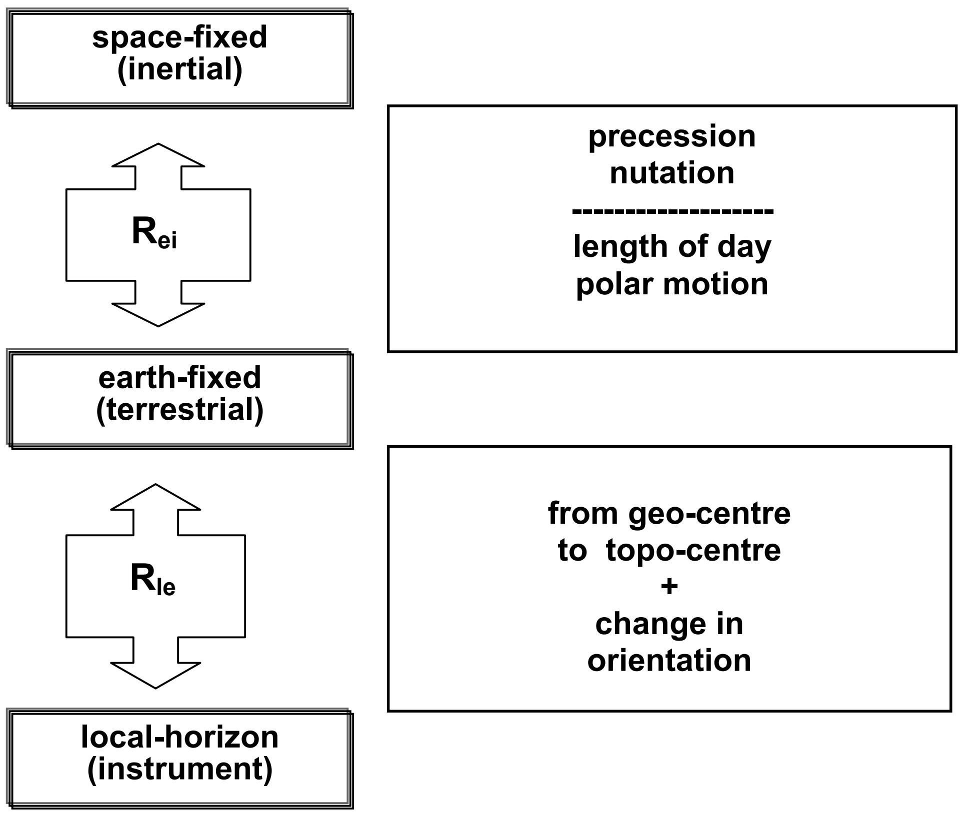

Figure 2.1: Hierarchy of reference systems - from space-fixed via earth-fixed to local horizontal

2.4 Special coordinate systems: In practice many additional coordinate systems are applied. Only two examples are given here:

- **Satellite orbit system $\mathbf{e}_o$, $o = 1, 2, 3$ ("o" for orbit)**:
  It is a triad rigidly tied to the momentary orientation of the orbit plane of a satellite. The orthonormal base vectors are

  $\mathbf{e}_{o=1}$ pointing towards the satellite perigee (closest point), see figure 4.1
  $\mathbf{e}_{o=3}$ perpendicular to the orbit plane
  $\mathbf{e}_{o=2}$ completing a right handed system.

- **Osculating orbit system $\mathbf{e}_s$, $s = 1, 2, 3$ ("s" for satellite)**:
  This triad is located at the centre of mass of the space craft with the following orthonormal base vectors:
  either
  $\mathbf{e}_{s=1}$ along track, in the direction of the velocity vector of the satellite,
  $\mathbf{e}_{s=2}$ perpendicular to the orbit plane (cross track),
  $\mathbf{e}_{s=3}$ completing a right-handed system (approximately radial)
  or
  $\mathbf{e}_{s=3}$ radial, from the earth's centre of mass,
  $\mathbf{e}_{s=2}$ perpendicular to the orbit plane (cross track)
  $\mathbf{e}_{s=1}$ completing a right handed system (approximately along track)

3. Transformation between coordinate systems

Each of the three hierarchy levels of reference systems, discussed in the previous chapter, takes an important role in positioning and navigation by space geodetic methods, the space-fixed, the earth-fixed as well as the local horizontal system. However, only if we know how to transform one into the other they become operational. Under the assumption of equal scale along each axis and in each of the coordinate systems considered here, transformation from one system into the other consists of a shift of origin from one system to the other system followed by a rotation between base vectors. (In reality it is all but trivial to warrant the same scale in all of our coordinate systems. After all each instrument carries its own scale and it is difficult to get all systems "calibrated".)

Let us consider the following situation, displayed in figure 3.1. The geocentre *O*, a terrestrial point *P* and satellite position *S* form a basic triangle in three-dimensional space; introducing the geocentric position vectors $\mathbf{r}_P$ and $\mathbf{r}_S$ and the topocentric position vectors $\mathbf{x}_S$ it can be expressed as:

$$\mathbf{r}_S = \mathbf{r}_P + \mathbf{x}_S . \tag{3.1}$$

All positioning and navigation by satellites and all orbit determination rests on this simple triangle condition between three fundamental vectors. However, behind this simplicity some complications are well hidden. For, although these three vectors are geometric objects invariant with respect to any chosen coordinate system, the components forming these three vectors are not. Each of the three vectors is represented in its own coordinate system.

(1) The geocentric orbit, i.e. vector $\mathbf{r}_S$, is given in the space fixed system $\mathbf{e}_i$.
(2) Terrestrial surface points, such as station or vehicle positions, i.e. vector $\mathbf{r}_P$, are best represented in the earth fixed system $\mathbf{e}_e$.

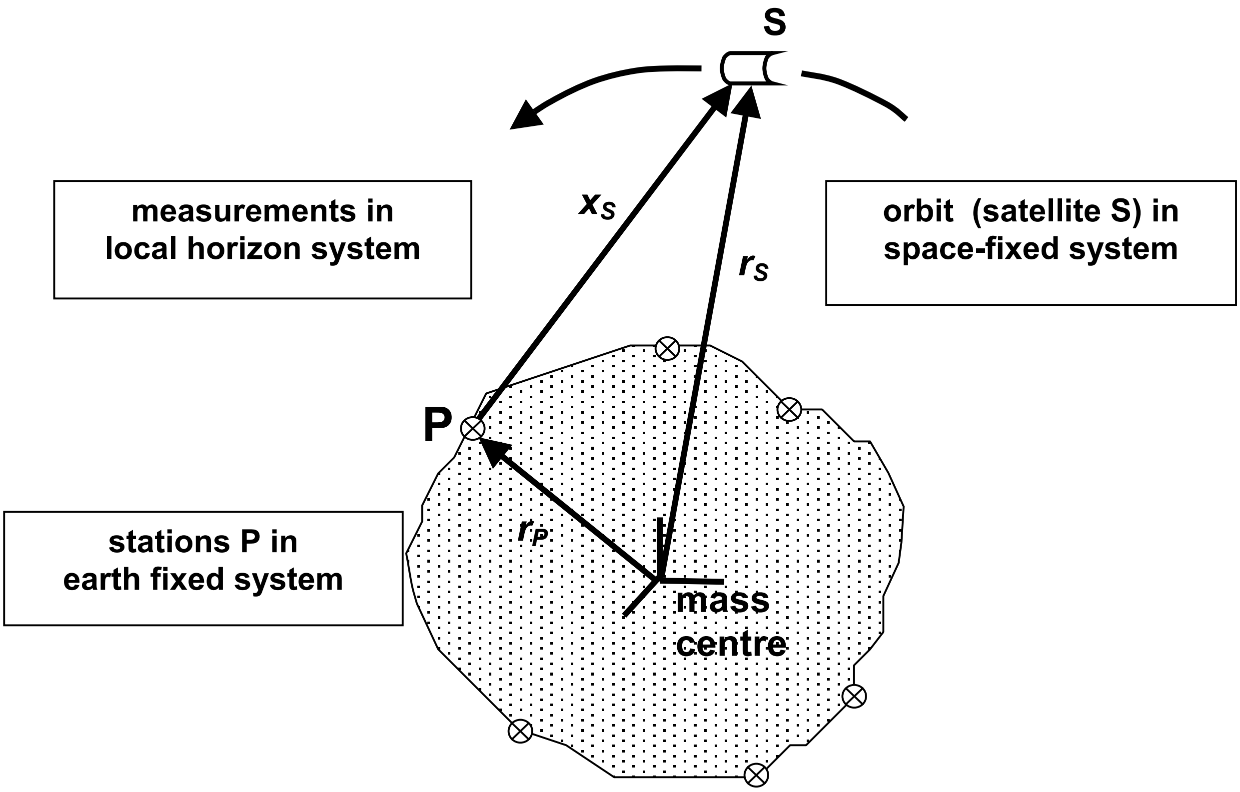

Figure 3.1: Vector sum of geocentric point position vector, topocentric measurement vector and geocentric satellite orbit vector, each one expressed in a different reference system.

Since we assumed that the origin of both, $\mathbf{e}_i$ and $\mathbf{e}_e$, is the geocentre, the two triads can be brought into coincidence by rotation only. Three independent elements are sufficient to perform this rotation. Thus, with the transformation R_e^i triad $\mathbf{e}_i$ can be rotated into the orientation $\mathbf{e}_e$ and vice versa:

$$\mathbf{e}_e = R_e^i \mathbf{e}_i \qquad \text{and} \qquad \mathbf{e}_i = R_i^e \mathbf{e}_e . \qquad (3.2\ \text{a, b})$$

Here the summation convention over repeated indices is applied; R_e^i and R_i^e can be represented by (3x3)-matrices containing the nine inner products (cosines) between the two sets of orthonormal base vectors. For their matrix representations $\mathbf{R}_{ei}$ and $\mathbf{R}_{ie}$ it holds:

$$\mathbf{R}_{ei} = \mathbf{R}_{ie}^{-1} = \mathbf{R}_{ie}^{T} .$$

The most common representation of rotations is that in terms of Eulerian transformation angles, where e.g. $R_1(\alpha)$ denotes a rotation about the 1-axis (*x*-axis) by the angle α in counter-clockwise direction.

The major contribution to R_e^i is the rotation of the earth about its spin axis with a period of 24^h; superimposed are precession, nutation, tiny variations in the steady angular velocity and polar motion. Only with R_e^i (or R_i^e) known, $\mathbf{r}_S$ and $\mathbf{r}_P$ can be transformed from $\mathbf{e}_i$ to $\mathbf{e}_e$ and back.

(3) Measurements, and therefore $\mathbf{x}_S$ too, are preferably represented in the local horizon system $\mathbf{e}_l$ located at the instrument:

$$\mathbf{x}_S = x_S^l \mathbf{e}_l$$

with time variable components x_S^l, l = 1, 2, 3.

The triads $\mathbf{e}_l$ and $\mathbf{e}_e$ are connected to each other by

$$\mathbf{e}_l = R_l^e \mathbf{e}_e \qquad \text{and} \qquad \mathbf{e}_e = R_e^l \mathbf{e}_l \,. \tag{3.3 a, b}$$

Since the *y*-axis of $\mathbf{e}_l$ (direction east) lies in a plane parallel to the equator, i.e. the {*x, y*}-plane of $\mathbf{e}_e$, only two independent angles are needed to rotate $\mathbf{e}_e$ into $\mathbf{e}_l$. These two angles are the astronomical latitude Φ and longitude Λ (or, as substitutes, the geographical latitude B and longitude L or the spherical latitude φ and longitude λ). The angles Φ and Λ define the plumb line direction at P in the coordinate system $\mathbf{e}_e$. They can be regarded as fixed in time.

We take now a closer look into the transformations R_e^i and R_l^e.

Transformation between space-fixed and earth-fixed system: In principle, only three Euler angles would suffice to express R_e^i. For phenomenological reasons it is however common practice to separate R_e^i into four parts

precession	$P = R_3(-z_A)R_2(\theta_A)R_3(-\zeta_A)$
nutation	$N = R_1(-\varepsilon-\Delta\varepsilon)R_3(-\Delta\psi)R_1(\varepsilon)$
earth rotation	$R_3(\Theta)$ with Θ = Greenwich apparent sidereal time (GAST)
polar motion	$W = R_2(-x_p)R_1(-y_p)$.

This means the transformation of the base vectors $\mathbf{e}_i$ of the space fixed system to the base vectors $\mathbf{e}_e$ of the earth fixed system is composed of

$$\begin{aligned} \mathbf{e}_e &= R_e^i \mathbf{e}_i \\ &= \left(W \cdot R_3(\Theta) \cdot N \cdot P\right)_e^i \; \mathbf{e}_i \end{aligned} \tag{3.4}$$

The equatorial **precession** parameters are (compare figure 3.2):

$$\zeta_A = 2306.''2181 \cdot t + 0.''30188 \cdot t^2 + 0.''017988 \cdot t^3$$

$$\theta_A = 2004.''3109 \cdot t - 0.''42665 \cdot t^2 - 0.''041833 \cdot t^3$$

$$z_A = 2304.''2181 \cdot t + 1.''09468 \cdot t^2 + 0.''018203 \cdot t^3$$

where the time is given in Julian centuries referred to $J2000.0\ (= 2451545.0)$:

$t = (T - J2000.0)/36525.0\,.$ (T is Julian Date)

The **nutation** part is more complicated. Several rather complex nutation models exist, originally based on a model of a rigid earth, meanwhile taking its elasticity into account. Corrections to these models are derived from VLBI and for short periodic terms (< 20 days) improvements are expected from GPS.
The mean obliquity of the ecliptic ε is:

$$\varepsilon = 23°26'21."488 - 46."8150 \cdot t - 0."000\,59 \cdot t^2 + 0."001813 \cdot t^3$$

The correction angles $\Delta\varepsilon$ and $\Delta\psi$ are expressed as series, see (MCCARTHY, 1996) and compare figure 3.3. An approximation, accurate to about 11", is given in (SEIDELMANN, 1992):

$$\Delta\psi = -0.°0048 \sin(125.°0 - 0.°05295 \cdot d) - 0.°0004 \sin(200.°9 + 1.°97129 \cdot d)$$

$$\Delta\varepsilon = 0.°0026 \cos(125.°0 - 0.°05295 \cdot d) + 0.°0002 \cos(200.°9 + 1.°97129 \cdot d)$$

where d is the number of day from $J2000.0$.

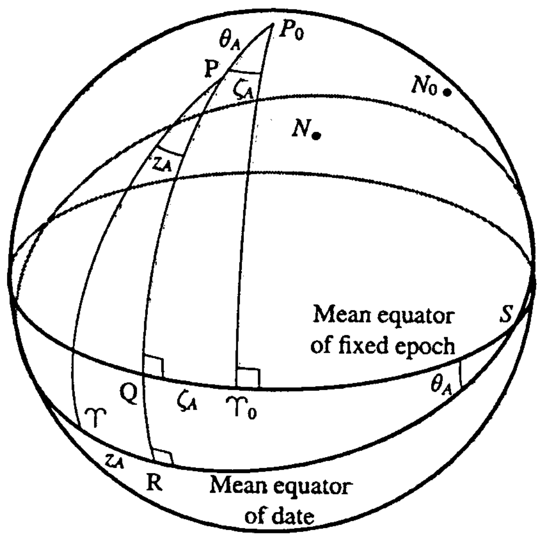

Figure 3.2: The precession angles ζ_A, z_A and θ_A
(from: SEIDELMANN, 1992, fig. 3.21.2)

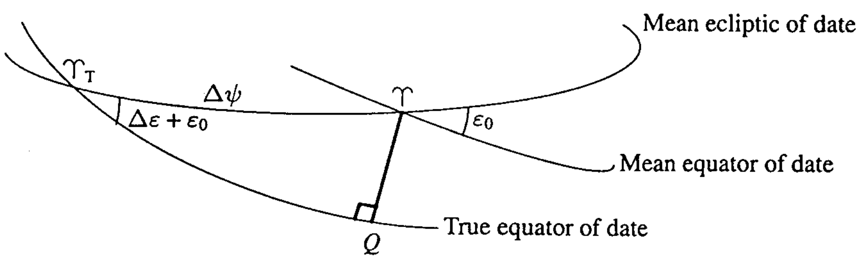

Figure 3.3: The nutation angles $\varepsilon + \Delta\varepsilon$, $\Delta\psi$ and θ_A
(from: SEIDELMANN, 1992, fig. 3.222.1)

The most important component of transformation R_e^i is $R_3(\Theta)$. It also defines the interface between space-fixed and earth-fixed. The angle Θ is Greenwich Apparent (true) Sidereal Time (GAST) which is the hour angle expressed in units of time between the Greenwich meridian at epoch and vernal equinox ♈. Consequently all deviations of the **earth's spin rate** from a constant angular motion are contained in Θ. There is a long-term deceleration very likely caused by tidal friction. All additional deviations are of the order of magnitude of a few milliseconds per day. Among others these fluctuations are caused by the tides of sun and moon, by exchange of angular momentum between atmosphere, oceans and solid earth and by long periodic effects such as postglacial mass readjustment. They are denoted as changes in length of day (LOD). See also figure 3.4.

Polar motion is defined as the movements of the earth's instantaneous spin axis relative to ITRF, i.e. to $\mathbf{e}_e$. It is split into a *x*- and *y*-component in a coordinate system fixed to the IERS reference pole (IRP) as a tangent plane. The *x*-axis points towards the Greenwich meridian, the *y*-axis towards the meridian $\lambda = 90°$. Polar motion is almost a circular motion superimposed by small fluctuations, compare figure 3.4. Its most prominent component is the Chandler period (free rotation of an elastic earth with a fluid core; the well-known Euler period would be the corresponding effect of the free gyroscopic motion of a solid earth). The exact mechanisms leading to the Chandler period are not yet completely understood. The Chandler motion is complemented by annual variations, due to the interaction of the atmosphere with the solid earth. Typical amplitudes are about $0.''3$ or about 8 m on the earth's surface. Both, polar motion and changes in length of day cannot be represented very well by models. They are derived from the various geodetic space techniques and represent a contribution of space geodesy to earth sciences.

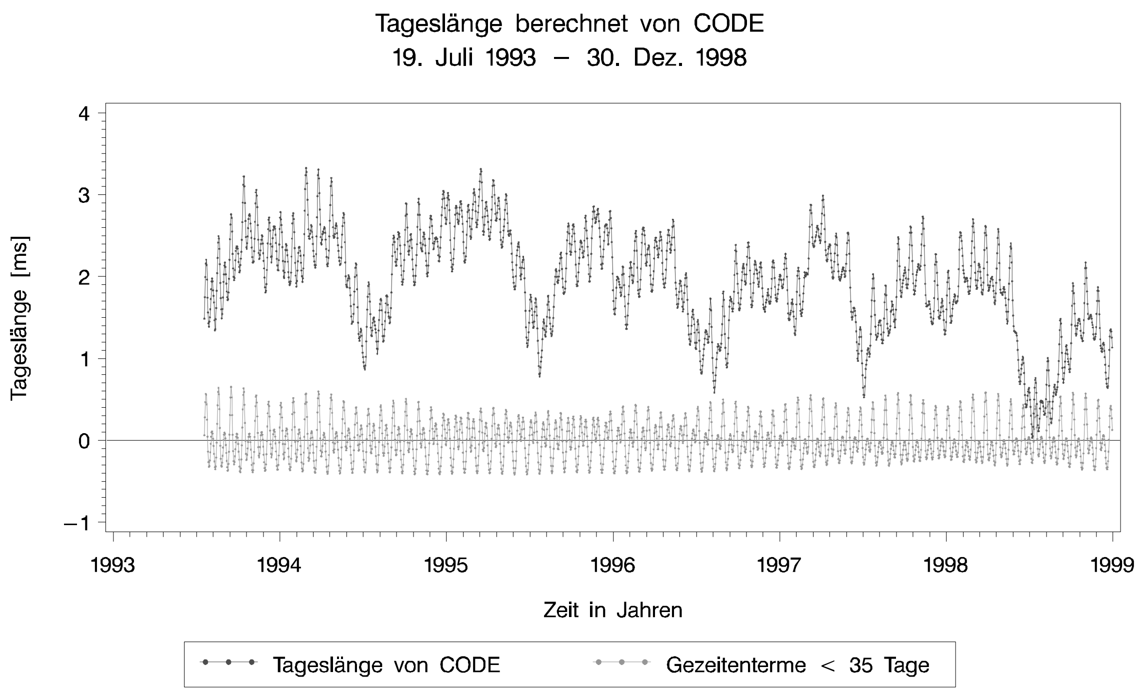

Figure 3.4: Variations in length of day between 1993 and 1999

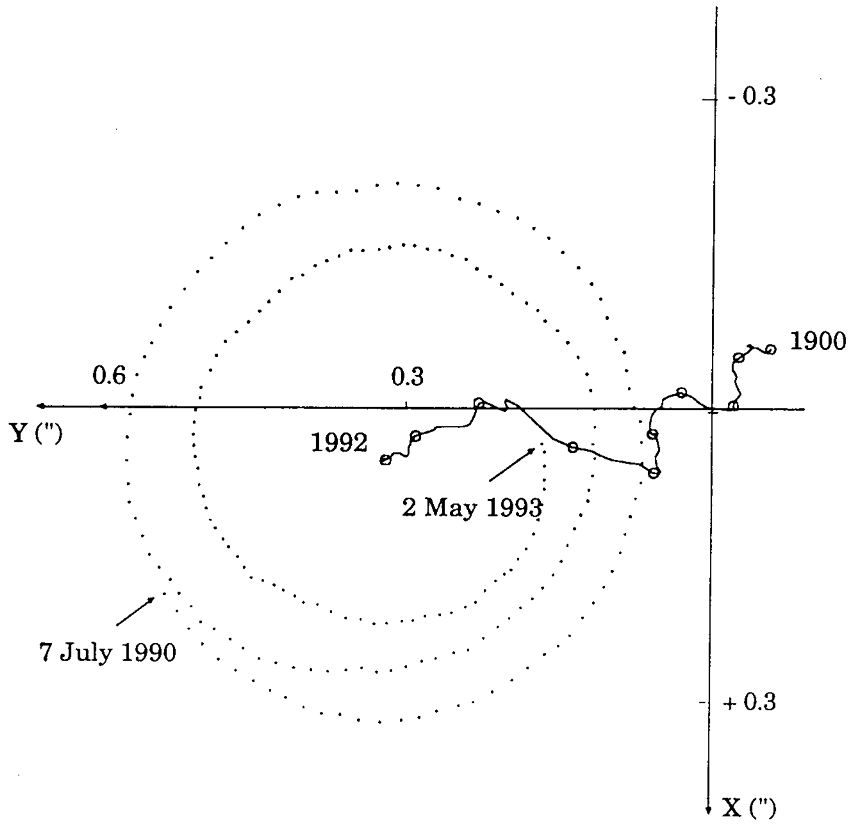

Figure 3.4: Secular polar wander between 1900 and 1992 and polar motion between 7 July 1990 and 2 May 1993

Transformation between earth-fixed and local-horizontal: The shift of origin from the geocentre to the topocentre is expressed by the position vector $\mathbf{r}_P$. The change in orientation between $\mathbf{e}_e$ and $\mathbf{e}_l$ is derived from

$$\begin{aligned}\mathbf{e}_l &= R_l^e \mathbf{e}_e \\ &= \left(Q_1 R_2\left(\frac{\pi}{2} - \Phi\right) R_3(\Lambda)\right)_l^e \mathbf{e}_e \end{aligned} \tag{3.5}$$

Thereby it is Q_1 the transformation from a left-handed to a right-handed system, in matrix form

$$\mathbf{Q}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The complete transformation takes the form

$$\mathbf{R}_{le} = \begin{pmatrix} -\sin\Phi\cos\Lambda & -\sin\Phi\sin\Lambda & \cos\Phi \\ -\sin\Lambda & \cos\Lambda & 0 \\ \cos\Phi\cos\Lambda & \cos\Phi\sin\Lambda & \sin\Phi \end{pmatrix} \tag{3.6}$$

with astronomical latitude Φ and astronomical longitude Λ. In the case of ellipsoidal or spherical approximation $\{\Phi, \Lambda\}$ are replaced by the corresponding geographical or spherical coordinates, $\{B, L\}$ and $\{\varphi, \lambda\}$, respectively.

Transformation between cartesian and geographical coordinates: In practice, in earth-fixed system $\mathbf{e}_e$, instead of cartesian coordinates x_P^e often geographical coordinates are used.

If we denote x_P^e by $\{X, Y, Z\}$ and geographical latitude, longitude and height above the reference ellipsoid by $\{B, L, h\}$ the following relations hold.

Forward Computation from $\{B, L, h\}$ to $\{X, Y, Z\}$:

$$
\begin{aligned}
X &= (N'+h)\cos B \cos L \\
Y &= (N'+h)\cos B \sin L \\
Z &= (N'-e^2 N'+h)\sin B
\end{aligned}
\qquad (3.7\text{ a - c})
$$

with normal curvature

$$N' = \frac{a}{\sqrt{1-e^2\sin^2 B}}$$

$$e^2 = \frac{a^2-b^2}{a^2}, \qquad e'^2 = \frac{a^2-b^2}{b^2}$$

with a and b length of the semi-major and semi-minor axis of the reference ellipsoid.

Backward Computation from $\{X, Y, Z\}$ to $\{B, L, h\}$:

(Method by Bowring)

$$\tan L = Y / X \qquad (3.8\text{ a - c})$$

auxiliary quantities:

$$p = \sqrt{X^2 + Y^2}$$

$$\tan\mu = \frac{Za}{pb}$$

$$\tan B = \frac{Z + e'^2 \sin^3\mu}{p - e^2 a \cos^3\mu}$$

$$h = p\cos B + Z\sin B - a\sqrt{1-e^2\sin^2 B}$$

4. Time

In the context of this lecture time is regarded as absolute and independent of space. In reality time keeping, and satellite measurements in general, are so accurate nowadays that adequate modelling requires the use of special and general theory of relativity for practical reason.

Time keeping requires a **periodic process**, a **counter** (in order to count the number of periods) and an **origin** where the counting starts. In addition, in order to be able to keep the same time at different locations **some means of transfer/transport of time** has to be available. There exist a number of natural "clocks" that produce very stable periodic oscillations: the orbit of the earth about the sun, of the moon about the earth and earth rotation. Their fundamental periods, year, month and day, are closely related to natural processes such as seasons that affect our living conditions and these periods define the basic structure of our life. From these fundamental periods the basic long term counting structure has been deduced, our **calendars** (we use the Gregorian calendar, adopted in 1582). In scientific work a continuous counting is preferable to the complicated structure of counting with months or year of varying length. For this purpose the **Julian date (JD)** has been invented with 36525 days per century.

The adopted reference date is

J2000.0 = 2000 Jan 1.5 = January 1, 2000 at 12^h

where it is

JD 2451 545.0.

For a long time the natural period day, and even more the revolution of the moon, were superior in terms of stability to any artificial clock. Only with the advent of quarz and atomic oscillators artificial clocks were created that meanwhile surpassed the precision and stability of natural clocks. Our current definition of the unit of second is based on the oscillation period of a caesium clock. In 1984 **atomic time** (Temps Atomique International = TAI) has been introduced as official, internationally adopted time. Its has a constant off-set of $32.^s184$ with respect to the **terrestrial dynamic time** (TDT). The latter is derived from models of planetary motion and based on the theory of relativity. TAI has a constant off-set of 19^s with respect to **GPS-time**.

Civilian time is related to the rhythm of day and night, i.e. to the rise and fall of the sun. Because of the complicated deviations of the apparent motion of the sun, some model or mean solar motion has been conceived. It refers to the Greenwich meridian and is denoted **universal time (UT)**. From UT standard zonal times have been deduced; in our case **MEZ**. Earth rotation - and therefore UT - exhibits a drift and small irregular fluctuations (changes in LOD) with respect to TAI. In order to circumvent this, a coordinated universal time **(UTC)** has been conceived, which on the one hand is kept synchronous with respect to TAI and on the other hand, through regular corrections (leap seconds), is kept within small bounds to follow the actual angular rate of the earth. The actual and uncorrected universal time is denoted **UT1**. It represents the actual phase angle of the rotating earth. The difference UT1-UTC is provided in monthly tables (see (McCarthy, 1996)) and provided as coded message in broadcasted time signals. If the difference UT1-UTC exceeds the size of 0.9^s a leap second is introduced. **UT0** completes the system of universal times. It contains all variations in rotation due to polar motion.

Finally, **sidereal time** is the angle of a terrestrial meridian (rotating with the earth) with respect to vernal equinox. The most prominent types of sidereal time are

Greenwich Mean Sidereal Time (GMST) $\overline{\Theta}$

and

Greenwich Apparent (or true) Sidereal Time (GAST) Θ.

GMST is corrected for fluctuations caused by nutation. It is

$$\begin{aligned}\Theta &= \overline{\Theta} + \Delta n \\ &= \overline{\Theta} + \Delta\psi \cdot \cos\varepsilon + 0.''00264 \cdot \sin\Omega + 0.''000063 \cdot \sin 2\Omega\end{aligned} \tag{4.1}$$

and Ω the mean node of the moon. Greenwich Apparent Sidereal Time is needed for the transformation from earth-fixed to space-fixed. The complete chain of time transformations is summarized in figure 4.1.

Since sidereal time is measured with respect to vernal equinox, i.e. the *x*-axis of $\boldsymbol{e}_i$, while universal time is a solar time and counted with respect to the apparent pass of the sun through the meridian at Greenwich the length of the year is different by one day: tropical year

in solar days:	365.24220
in sidereal days:	366.24220.

This difference has to be accounted for when transforming UT1 to GMST. It holds

$$GMST = UT1 + \alpha(\Theta) - 12^h \tag{4.2}$$

with the right ascension of the sun:

$$\alpha(\Theta) = 12^h + \left(24110.^s54841 + 8640.^s812866 \cdot t + 0.^s093104 \cdot t^2 - 6.^s2 \cdot 10^{-6} \cdot t^3\right) \tag{4.3}$$

and

$$t = (T - J2000.0)/36525.0$$

with T Julian Date at epoch and *J2000.0 = JD2451545.0.*

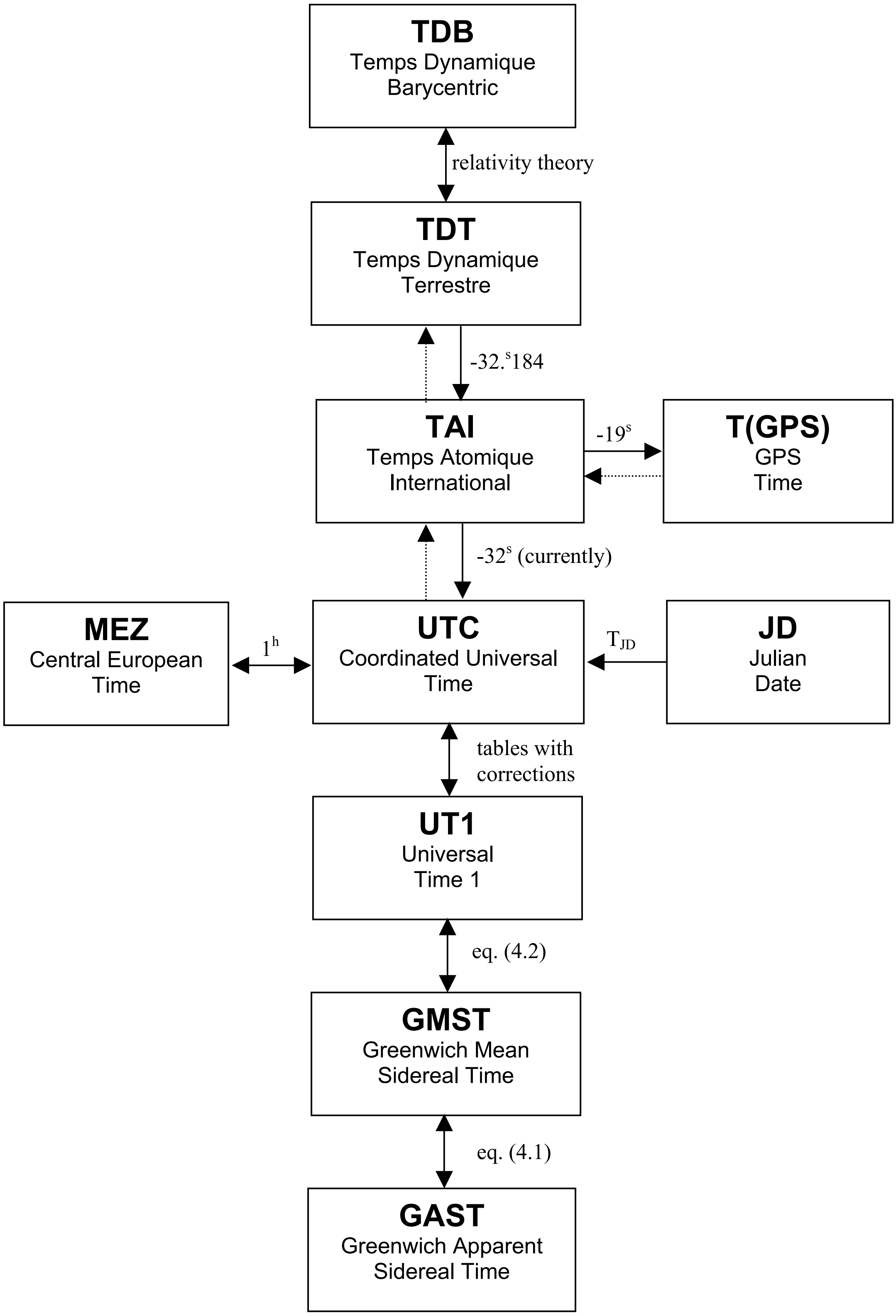

Figure 4.1: Transformation between time systems (after: MÜLLER, 1999)

Tutorial - Representation of Satellite Orbits in Various Coordinate Systems

The purpose of this tutorial is to give an introduction into the representation of satellite orbits in various, often used coordinate systems. A collection of related formulas is given in table 5.1.
We consider the most simple type of satellite orbit, a Kepler ellipse. Its shape is defined by a and b. The earth is assumed to be a perfect and homogeneous sphere with its centre in one of the focal points of the ellipse. In the orbit coordinate system $\mathbf{e}_o$ the position vector of the satellite takes the form (compare figure 5.1):

$$\mathrm{r}_S = \begin{pmatrix} r_S \cos\nu \\ r_S \sin\nu \\ 0 \end{pmatrix}_o = \begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix}_o \tag{5.1}$$

Since $\mathbf{e}_{o=3}$ is perpendicular to the orbit plane the *z*-coordinate of $\mathbf{r}_S$ is zero. Both the true anomaly ν and the eccentric anomaly E are functions of time, as is the radial distance r_s. Eq. (5.1) allows us to display the Kepler ellipse in the {*x, y*}-plane of $\mathbf{e}_o$, i.e. to show all satellite positions as a function of time.

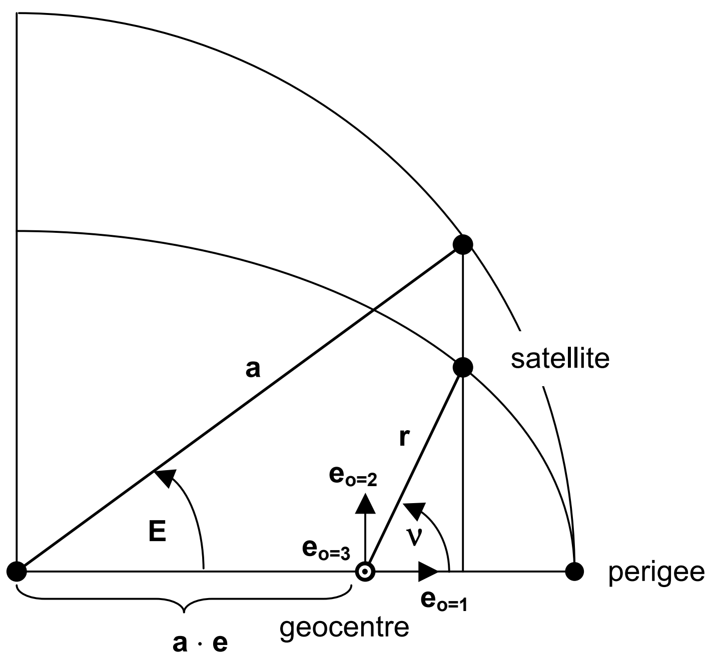

Figure 5.1: Orbital ellipse

In the space-fixed triad $\mathbf{e}_i$ the same position vector takes the form

$$\mathbf{r}_s = \begin{pmatrix} r_s \cos\delta\cos\alpha \\ r_s \cos\delta\sin\alpha \\ r_s \sin\delta \end{pmatrix}_i , \tag{5.2}$$

with direction angles right ascension α and declination δ.

Table 5.1: Collection of formulas

ellipse:

semi-major axis	a
semi-minor axis	b
first eccentricity	$e = \sqrt{\frac{a^2 - b^2}{a^2}}$
second eccentricity	$e' = \sqrt{\frac{a^2 - b^2}{b^2}}$

position in orbital plane (compare figure 5.1):

orbit period	T	
mean motion	$n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}}$	(mean angular velocity)
mean anomaly	$l = n \cdot (t - t_o)$	
eccentric anomaly	$E - e \sin E = l$	(l counted from perigee)
true anomaly	$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$	or
	$\tan \nu = \frac{\sqrt{1-e^2} \sin E}{\cos E - e}$	

$$r_S = \frac{a(1-e^2)}{1 + e \cos \nu}$$

$$r_S = a(1 - e \cos E)$$

gravitational constant · mass of the earth: GM = 398 600.5 km³/s²

orbit elements of the two GPS-satellites shown in the figures (2 revolutions):

$T \cong 11^h 58^m$

$\Omega = 0°$ and $60°$

$\omega = 0°$

$i = 55°$

$a = 26560$ km

$e = 0.01$

Orbit and hierarchy of coordinate systems

In order to be able to display the Kepler ellipse in $\mathbf{e}_i$ we have to know the transformation from the orbit system to the space-fixed system, R_i^o. As long as the earth is a sphere, the orbit ellipse remains fixed in space, no precession of the orbital plane occurs. Thus only the three time invariant rotation angles from $\mathbf{e}_o$ to $\mathbf{e}_i$ are needed. They are the ascending node Ω, the inclination i of the orbit plane with respect to the equator and the argument of perigee ω, compare figure 5.2:

$$\begin{aligned}\mathbf{e}_i &= R_i^o \mathbf{e}_o \\ &= \left(R_3(-\Omega)R_1(-i)R_3(-\omega)\right)_i^o \mathbf{e}_o\end{aligned} \tag{5.3}$$

or in matrix form

$$\mathbf{R}_{io} = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{pmatrix} \tag{5.4}$$

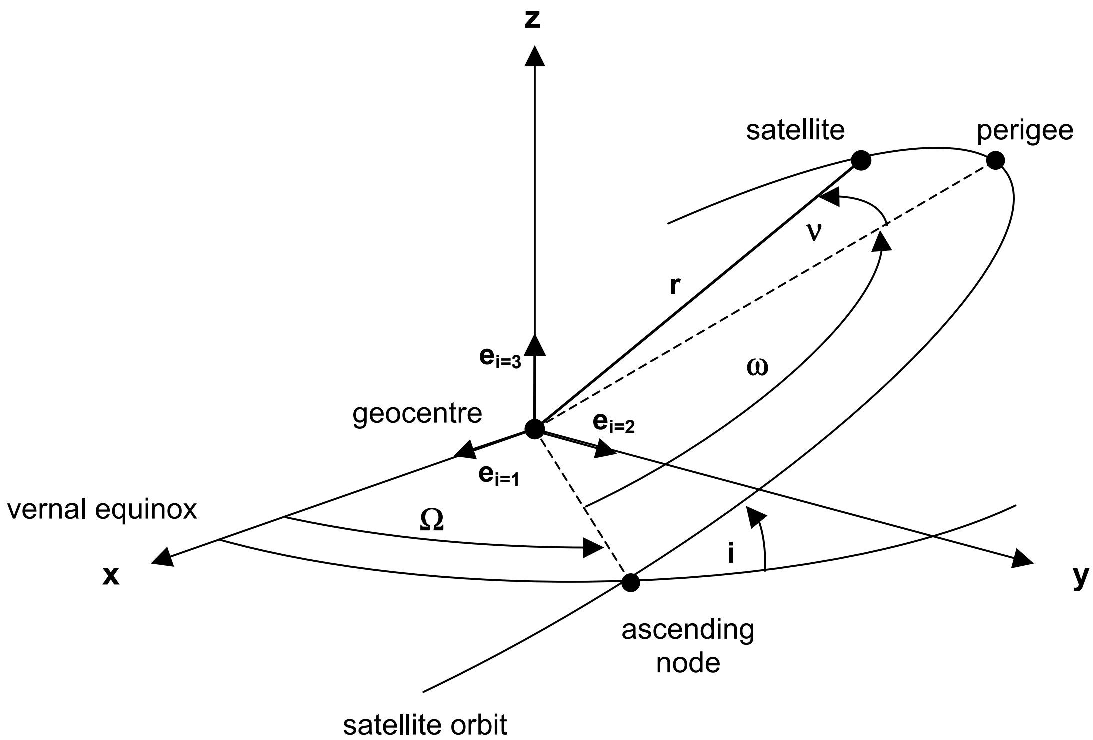

Figure 5.2: Orbital orientation in a space-fixed coordinate system

With R_i^o the orbit can now be displayed in $\mathbf{e}_i$; a relation with right ascension and declination (α, δ) is established. The result is shown in figure 5.3 for GPS-type satellite orbits. The orbit elements are given in table 5.1.

It is more interesting to connect the orbit triad via $\mathbf{e}_i$ with the earth-fixed triad $\mathbf{e}_e$ and the local horizon system $\mathbf{e}_l$. If all small effects, such as precession, nutation, changes in length of day, and polar motion are neglected, e.q. (3.4) becomes

$$\begin{aligned}\mathbf{e}_e &= R_e^i \mathbf{e}_i \\ &= \left(R_3(\Theta)\right)_e^i \mathbf{e}_i\end{aligned} \tag{5.5}$$

with matrix representation

$$\mathbf{R}_{ei} = \begin{pmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.6}$$

where Greenwich apparent sideral time Θ can be replaced by

$$\Theta = n(t - t_o).$$

Now we find for the components of $\mathbf{r}_s$ in $\mathbf{e}_e$:

$$\mathbf{r}_S = \begin{pmatrix} r_S^{e=1} \\ r_S^{e=2} \\ r_S^{e=3} \end{pmatrix} = r_S^i R_i^e r_S \begin{pmatrix} \cos\delta\cos(\Theta-\alpha) \\ -\cos\delta\sin(\Theta-\alpha) \\ \sin\delta \end{pmatrix}_e . \tag{5.7}$$

The result is shown in figure 5.4.

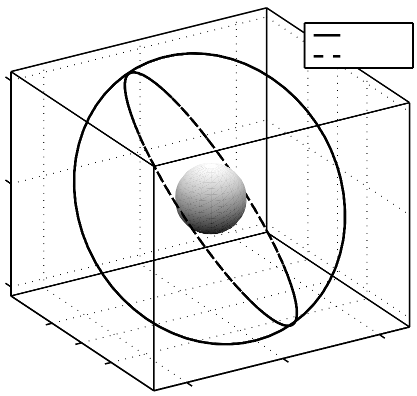

Figure 5.3: The orbits of two GPS satellites in a space-fixed geocentric coordinate system

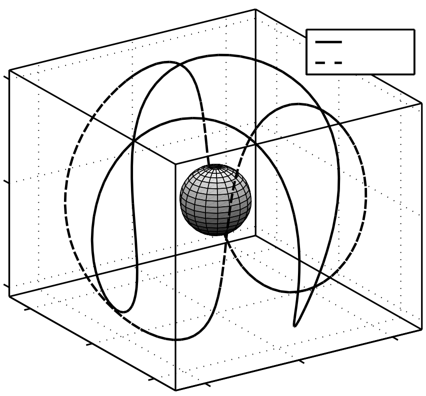

Figure 5.4: The orbits of two satellites in an earth-fixed geocentric coordinate system

Finally, the same orbit is brought into the local horizontal triad $\mathbf{e}_l$. Since we are only interested in the direction of the satellite, the shift of origin can be disregarded.
In terms of the orientation angles in $\mathbf{e}_l$, azimuth and zenith distance, it takes the simple form:

$$\mathbf{r}_S = r_S \begin{pmatrix} \sin z \cos A \\ \sin z \sin A \\ \cos z \end{pmatrix}_l \tag{5.8}$$

and using R_i^e, eq. (3.6), applied to (5.7) we arrive at

$$\mathbf{r}_S = r_S \begin{pmatrix} \cos\varphi \sin\delta - \sin\varphi \cos\delta \cos\tau \\ -\cos\delta \sin\tau \\ \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos\tau \end{pmatrix}_l . \tag{5.9}$$

We replaced astronomical latitude and longitude, Φ and Λ, by the spherical ones, φ and λ. The angle τ is the so-called hour angle

$$\tau = \Theta + \lambda - \alpha . \tag{5.10}$$

Comparison of (5.8) and (5.9) yields A and z as a function of δ, φ and τ:

$$\begin{aligned} \cot A &= -\frac{\cos\Phi \tan\delta - \sin\Phi \cos\tau}{\sin\tau} , \\ \cos z &= \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos\tau . \end{aligned} \tag{5.11}$$

In figure 5.5 azimuth and zenith distance are shown in stereographic projection, with the projection plane coinciding with the horizon and centered at station P. This type of plot is denoted **sky plot** or **visibility plot**. It provides a convenient picture of the rise of a satellite above the horizon and of its fall. It also shows under what elevation and azimuth a satellite passes a station.

Figure 5.4, the orbit of a satellite in $\mathbf{e}_e$, is not very instructive. Instead, one often computes from the cartesian coordinates, eq. (5.7), the ellipsoidal coordinates $\{B, L\}$ using eq. (3.8) or simple the spherical coordinates $\{\varphi, \lambda\}$. They can be shown as a so-called **ground track plot**, i.e. as a projection of $\mathbf{r}_s$ onto the sphere. Figure 5.6 gives the ground track of the two GPS satellites.

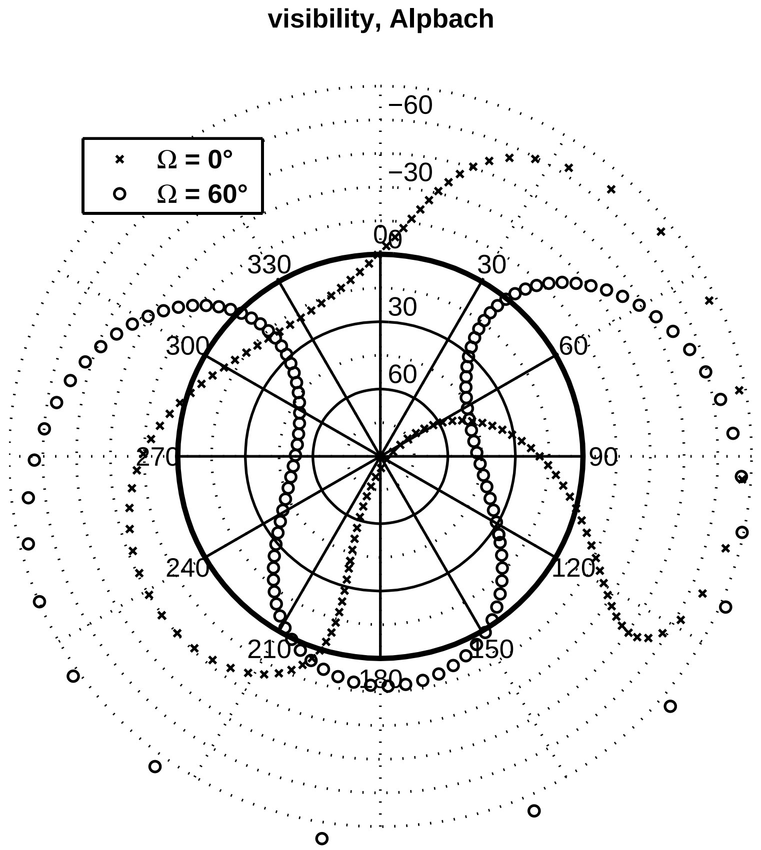

Figure 5.5: Sky plot or visibility plot, centered at Alpbach ($\varphi = 47°24'$ N, $\lambda = 11°57'$ E), with traces of two GPS satellites

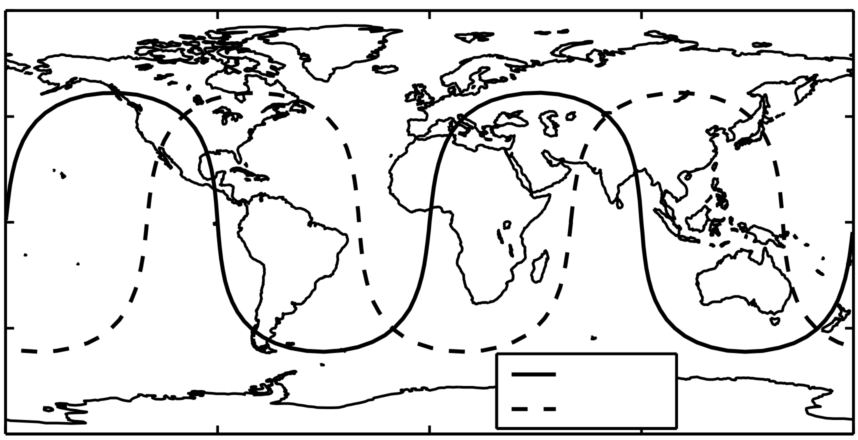

Figure 5.6: Ground tracks of two GPS satellites

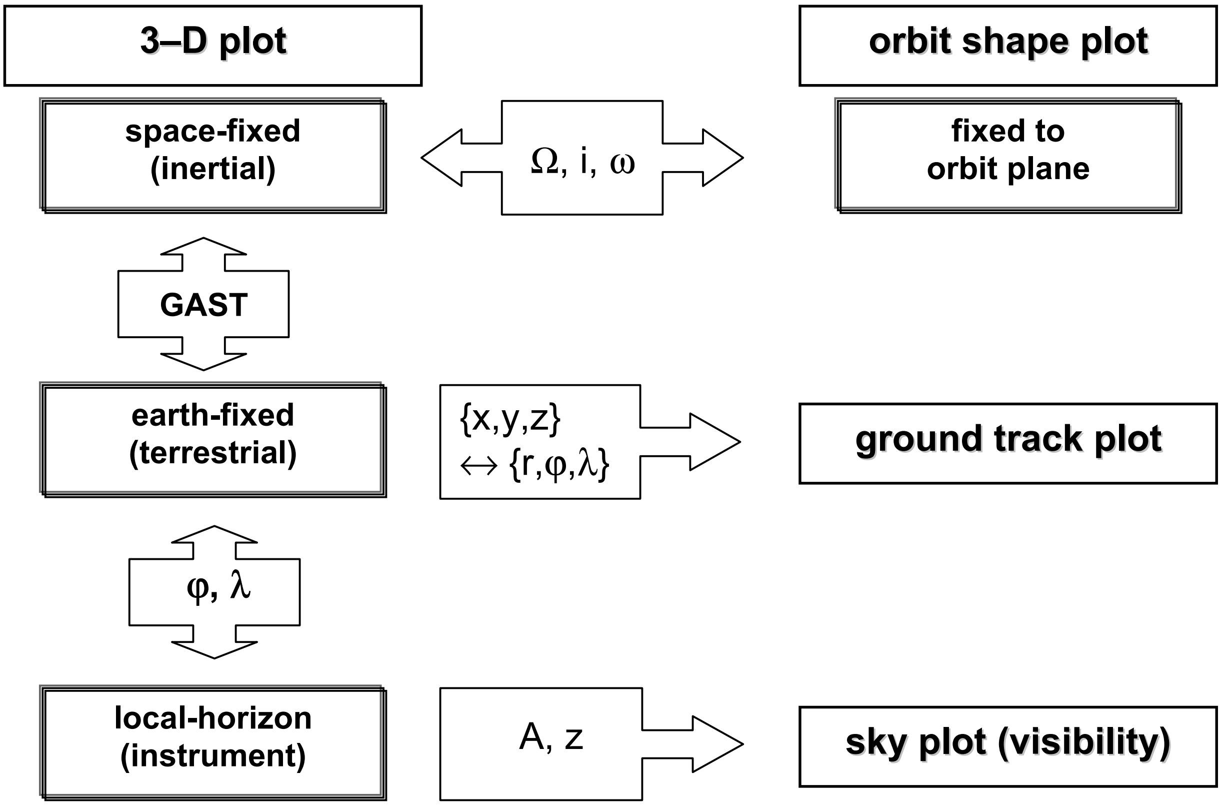

Figure 5.7: Hierarchy of reference systems and the associated graphical representations

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