9 Statistical methods in physical geodesy

9.1 Introduction

Some of the most important problems of gravimetric geodesy are formulated and solved in terms of integrals extended over the whole earth. An example is Stokes' formula. Thus, in principle, we need the gravity g at every point of the earth's surface. As a matter of fact, even in the densest gravity net we measure g only at relatively few points so that we must estimate g at other points by *interpolation*. In large parts of the oceans we have made no observations at all; these gaps must be filled by some kind of *extrapolation*.

Mathematically, there is no difference between interpolation and extrapolation; therefore they are denoted by the same term, *prediction*.

Prediction (i.e., interpolation or extrapolation) cannot give exact values; hence, the problem is to estimate the errors that are to be expected in the gravity g or in the gravity anomaly Δg. As usual, gravity disturbances δg are appropriately comprised whenever we speak of gravity anomalies.

Since Δg is further used to compute other quantities, such as the geoidal undulation N or the deflection components ξ and η, we must also investigate the influence of the prediction errors of Δg on N, ξ, η, etc. This is called *error propagation*, which will play a basic role.

It is also important to know which prediction method gives highest accuracy, either in Δg or in derived quantities N, ξ, η, etc. To be able to find these "best" prediction methods, it is necessary to have solved the previous problem, to know the prediction error of Δg and its influence on the derived quantities.

Summarizing, we have the following problems:

1. estimation of interpolation and extrapolation errors of Δg (or δg);
2. estimation of the effect of these errors on derived quantities (N, ξ, η, etc.);
3. determination of the best prediction method.

Since we are interested in the average rather than the individual errors, we are led to a statistical treatment. This will be the topic of the present chapter.

9.2 The covariance function

It is quite remarkable that all the problems mentioned above can be solved by means of only one function of one variable, without any other information. This is the *covariance function* of the gravity anomalies.

First we need a measure of the average size of the gravity anomalies Δg. If we form the average of Δg over the whole earth, we get the value zero:

$$M\{\Delta g\} \equiv \frac{1}{4\pi} \iint\limits_{\sigma} \Delta g \, d\sigma = 0 \, . \tag{9–1}$$

The symbol M stands for the average over the whole earth (over the unit sphere); this average is equal to the integral over the unit sphere divided by its area 4π. The integral is zero if there is no term of degree zero in the expansion of the gravity anomalies Δg into spherical harmonics, that is, if a reference ellipsoid of the same mass as the earth and of the same potential as the geoid is used. This will be assumed throughout this chapter.

Note that if this is not the case, that is, if $M\{\Delta g\} = m \neq 0$, then we may form new gravity anomalies $\Delta g^* = \Delta g - m$ by subtracting the average value m. Then $M\{\Delta g^*\} = 0$ and all the following developments apply to the "centered" anomalies Δg^*.

Clearly, the quantity $M\{\Delta g\}$, which is zero, cannot be used to characterize the average size of the gravity anomalies. Consider then the average square of Δg,

$$\mathrm{var}\{\Delta g\} \equiv M\{\Delta g^2\} = \frac{1}{4\pi} \iint\limits_{\sigma} \Delta g^2 \, d\sigma \, . \tag{9–2}$$

It is called the *variance* of the gravity anomalies. Its square root is the *root mean square (rms) anomaly*:

$$\mathrm{rms}\{\Delta g\} \equiv \sqrt{\mathrm{var}\{\Delta g\}} = \sqrt{M\{\Delta g^2\}} \, . \tag{9–3}$$

The rms anomaly is a very useful measure of the average size of the gravity anomalies; it is usually given in the form

$$\mathrm{rms}\{\Delta g\} = \pm 35 \text{ mgal} \, ; \tag{9–4}$$

the sign $\pm$ expresses the ambiguity of the sign of the square root and symbolizes that Δg may be either positive or negative. The rms anomaly is very intuitive; but the variance of Δg is more convenient to handle mathematically and admits an important generalization.

Instead of the average square of Δg, consider the average product of the gravity anomalies $\Delta g\,\Delta g'$ at each pair of points P and P' that are at a constant distance s apart. This average product is called the *covariance* of the gravity anomalies for the distance s and is defined by

$$\operatorname{cov}_s\{\Delta g\} \equiv M\{\Delta g\,\Delta g'\}\,. \tag{9–5}$$

The average is to be extended over all pairs of points P and P' for which $PP' = s = \text{constant}$.

The covariances characterize the *statistical correlation* of the gravity anomalies Δg and $\Delta g'$, which is their tendency to have about the same size and sign. If the covariance is zero, then the anomalies Δg and $\Delta g'$ are uncorrelated or independent of one another (note that in the precise language of mathematical statistics, zero correlation and independence are not quite the same, but we may neglect the difference here!); in other words, the size or sign of Δg has no influence on the size or sign of $\Delta g'$. Gravity anomalies at points that are far apart may be considered uncorrelated or independent because the local disturbances that cause Δg have almost no influence on $\Delta g'$ and vice versa.

If we consider the covariance as a function of distance $s = PP'$, then we get the *covariance function* $C(s)$ mentioned at the beginning:

$$C(s) \equiv \operatorname{cov}_s\{\Delta g\} = M\{\Delta g\,\Delta g'\} \quad (PP' = s)\,. \tag{9–6}$$

For $s = 0$, we have

$$C(0) = M\{\Delta g^2\} = \operatorname{var}\{\Delta g\} \tag{9–7}$$

according to (9–2). The covariance for $s = 0$ is the variance.

A typical form of the function $C(s)$ is shown in Fig. 9.1. For small distances s (1 km, say), $\Delta g'$ is almost equal to Δg, so that the covariance is almost equal to the variance; in other words, there is a very strong correlation. The covariance $C(s)$ decreases with increasing s because then the

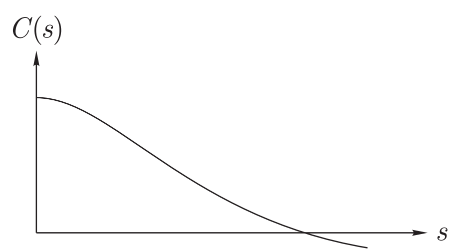

Fig. 9.1. The covariance function

anomalies Δg and $\Delta g'$ become more and more independent. For very large distances, the covariance will be very small but not in general exactly zero because the gravity anomalies are affected not only by local mass disturbances but also by regional factors. Therefore, we may expect an oscillation of the covariance between small positive and negative values.

Note that positive covariances mean that Δg and $\Delta g'$ tend to have the same size and the same sign; negative covariances mean that Δg and $\Delta g'$ tend to have the same size and opposite sign. The stronger this tendency, the larger is $C(s)$; the absolute value of $C(s)$ can, however, never exceed the variance $C(0)$.

The practical determination of the covariance function $C(s)$ is somewhat problematic. If we were to determine it exactly, we should have to know gravity at every point of the earth's surface. This we obviously do not know; and if we knew it, then the covariance function would have lost most of its significance because then we could solve our problems rigorously without needing statistics. As a matter of fact, we can only estimate the covariance function from samples distributed over the whole earth. But even this is not quite possible at present because of the imperfect or completely missing gravity data over the oceans. For a discussion of sampling and related problems see Kaula (1963, 1966 b).

The first comprehensive estimate of the covariance function was made by Kaula (1959). Some of his values are given in Table 9.1 for historical interest. They refer to free-air anomalies. The argument is the spherical distance

$$\psi = \frac{s}{R} \tag{9–8}$$

corresponding to a linear distance s measured on the earth's surface; R is a mean radius of the earth. The rms free-air anomaly is

$$\mathrm{rms}\{\Delta g\} = \sqrt{1201} = \pm\,35 \text{ mgal}\,. \tag{9–9}$$

We see that $C(s)$ decreases with increasing s and that, for $s/R > 30°$, very small values oscillate between plus and minus.

For some purposes we need a *local* covariance function rather than a global one; then the average M is extended over a limited area only, instead of over the whole earth as above. Such a local covariance function is useful for more detailed studies in a limited area – for instance, for interpolation problems. As an example we mention that Hirvonen (1962), investigating the local covariance function of the free-air anomalies in Ohio, found numerical values that are well represented by an analytical expression of the form

$$C(s) = \frac{C_0}{1 + (s/d)^2}\,, \tag{9–10}$$

Table 9.1. Estimated values of the covariance function for free-air anomalies [unit mgal^2]

ψ	$C(\psi)$	ψ	$C(\psi)$	ψ	$C(\psi)$
0.0°	+1201	8°	+124	27°	+18
0.5°	751	9°	104	29°	+6
1.0°	468	10°	82	31°	+8
1.5°	356	11°	76	33°	+5
2.0°	332	13°	54	35°	−8
2.5°	306	15°	47	40°	−12
3.0°	296	17°	45	50°	−20
4.0°	272	19°	34	60°	−30
5.0°	246	21°	35	90°	−4
6.0°	214	23°	10	120°	+12
7.0°	174	25°	20	150°	−21

where

$$C_0 = 337 \text{ mgal}^2 , \qquad d = 40 \text{ km} . \tag{9–11}$$

This function is valid for $s < 100$ km.

In the meantime it has been recognized that a proper determination of global and local covariance functions is a central practical problem in this context.

The Tscherning–Rapp covariance model and the COVAXN subroutine

The fundamental covariance model by Tscherning and Rapp (1974) and the subroutine COVAXN (Tscherning 1976) are still very much up to date, as the following quotation from Kühtreiber (2002 b) shows:

"The global covariance function of the gravity anomalies $C_g(P,Q)$ given by Tscherning and Rapp (1974, p. 29) is written as

$$C_g(P,Q) = A \sum_{n=3}^{\infty} \frac{n-1}{(n-2)(n+B)} \, s^{n+2} P_n(\cos\psi) , \tag{9–12}$$

where $P_n(\cos\psi)$ denotes the Legendre polynomial of degree n; ψ is the spherical distance between P and Q; and A, B and s are the model parameters. A closed expression for (9–12) is available in (ibid., p. 45).

The local covariance function of gravity anomalies $C(P, Q)$ given by Tscherning–Rapp can be defined as

$$C(P,Q) = A \sum_{n=N+1}^{\infty} \frac{n-1}{(n-2)(n+B)} \, s^{n+2} P_n(\cos\psi) \, . \tag{9–13}$$

Modeling the covariance function means in practice fitting the empirically determined covariance function (through its three essential parameters: the variance C_0, the correlation length ξ and the variance of the horizontal gradient G_0) to the covariance function model. Hence the four parameters A, B, N and s are to be determined through this fitting procedure. A simple fitting of the empirical covariance function was done using the COVAXN-subroutine (Tscherning 1976).

The essential parameters of the empirical covariance parameters for 2489 gravity stations in Austria are 740.47 mgal^2 for the variance C_0 and 43.5 km for the correlation length ψ_1. The value of the variance for the horizontal gradient G_0 was roughly estimated as 100 E^2 (note that E indicates the Eötvös unit, where 1 E = 10^{-9} s^{-2}).

With a fixed value $B = 24$, the following Tscherning–Rapp covariance function model parameters were fitted: $s = 0.997\,065$, $A = 746.002$ mgal^2 and $N = 76$. The parameters were used for the astrogeodetic, the gravimetric as well as the combined geoid solution." (End of quotation.)

The Tscherning–Rapp model can be summed to get closed expressions. Its popularity is due to its comprehensiveness: there are expressions for covariances of various quantities derived by covariance propagation (Sect. 10.1), and to its flexibility since it contains several parameters which can be given various numerical values.

Remark. The spherical-harmonic expression of the covariance function is considered in Sect. 9.3. The theory of global and local covariance functions is described in great detail in Moritz (1980 a: Sects. 22 and 23). The three essential parameters of a local covariance function (variance C_0, correlation length ξ, and curvature parameter G_0) are also defined there. Fundamental numerical studies on local covariance functions have been made by Kraiger (1987, 1988).

9.3 Expansion of the covariance function in spherical harmonics

The more or less complicated integral formulas of physical geodesy frequently take on a much simpler form if they are rewritten in terms of spherical harmonics. A good example is Stokes' formula (see Sect. 2.15).

Unfortunately, this theoretical advantage is in most cases balanced by the practical disadvantage that the relevant series converge very slowly. In certain cases, however, the convergence is good. Then the use of spherical harmonics is very convenient practically; we consider such a case in the next section.

The spherical-harmonic expansion of the gravity anomalies Δg may be written in different ways, such as

$$\Delta g(\vartheta, \lambda) = \sum_{n=2}^{\infty} \Delta g_n(\vartheta, \lambda) , \tag{9–14}$$

where $\Delta g_n(\vartheta, \lambda)$ is the Laplace surface harmonic of degree n; or, more explicitly,

$$\Delta g(\vartheta, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[a_{nm} \mathcal{R}_{nm}(\vartheta, \lambda) + b_{nm} \mathcal{S}_{nm}(\vartheta, \lambda) \right] , \tag{9–15}$$

where

$$\begin{aligned} \mathcal{R}_{nm}(\vartheta, \lambda) &= P_{nm}(\cos\vartheta) \cos m\lambda , \\ \mathcal{S}_{nm}(\vartheta, \lambda) &= P_{nm}(\cos\vartheta) \sin m\lambda \end{aligned} \tag{9–16}$$

are the conventional spherical harmonics; or in terms of fully normalized harmonics (see Sect. 1.10):

$$\Delta g(\vartheta, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[\bar{a}_{nm} \bar{\mathcal{R}}_{nm}(\vartheta, \lambda) + \bar{b}_{nm} \bar{\mathcal{S}}_{nm}(\vartheta, \lambda) \right] . \tag{9–17}$$

Here ϑ is the polar distance (complement of geocentric latitude) and λ is the longitude.

Let us now find the average products of two Laplace harmonics

$$\Delta g_n(\vartheta, \lambda) = \sum_{m=0}^{n} \left[\bar{a}_{nm} \bar{\mathcal{R}}_{nm}(\vartheta, \lambda) + \bar{b}_{nm} \bar{\mathcal{S}}_{nm}(\vartheta, \lambda) \right] . \tag{9–18}$$

These average products are

$$M\{\Delta g_n \Delta g'_n\} = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\vartheta=0}^{\pi} \Delta g_n(\vartheta, \lambda) \, \Delta g'_n(\vartheta, \lambda) \sin\vartheta \, d\vartheta \, d\lambda , \tag{9–19}$$

since the averaging is extended over the whole earth, that is, over the whole unit sphere. Take first $n' = n$, which gives the average square of the Laplace harmonic of degree n:

$$M\{\Delta g_n^2\} = \frac{1}{4\pi} \int_{\lambda=0}^{2\pi} \int_{\vartheta=0}^{\pi} \left[\Delta g_n(\vartheta, \lambda) \right]^2 \sin\vartheta \, d\vartheta \, d\lambda . \tag{9–20}$$

Substituting (9–18) and taking into account the orthogonality relations (1–83) and the normalization (1–91), we easily find

$$M\{\Delta g_n^2\} = \sum_{m=0}^{n} (\bar{a}_{nm}^2 + \bar{b}_{nm}^2) \,. \tag{9–21}$$

Consider now the average product (9–19) of two Laplace harmonics of different degree, $n' \neq n$. Owing to the orthogonality of the spherical harmonics, the integral in (9–19) is zero:

$$M\{\Delta g_n \Delta g_n'\} = 0 \quad \text{if} \;\; n' \neq n \,. \tag{9–22}$$

In statistical terms this means that two Laplace harmonics of different degrees are *uncorrelated* or, broadly speaking, *statistically independent.*

In a way similar to that used for the gravity anomalies, we may also expand the covariance function $C(s)$ into a series of spherical harmonics. Let us take an arbitrary, but fixed, point P as the pole of this expansion. Thus spherical polar coordinates ψ (angular distance from P) and α (azimuth) are introduced (Fig. 9.2). The angular distance ψ corresponds to the linear distance s according to (9–8). If we expand the covariance function, with argument ψ, into a series of spherical harmonics with respect to the pole P and coordinates ψ and α, we have

$$C(\psi) = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[c_{nm} \mathcal{R}_{nm}(\psi, \alpha) + d_{nm} \mathcal{S}_{nm}(\psi, \alpha)\right] , \tag{9–23}$$

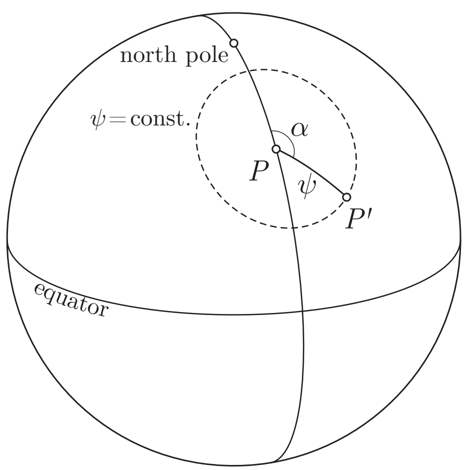

Fig. 9.2. Spherical coordinates ψ, α

which is of the same type as (9–15). But since C depends only on the distance ψ and not on the azimuth α, the spherical harmonics cannot contain any terms that explicitly depend on α. The only harmonics independent of α are the zonal functions

$$\mathcal{R}_{n0}(\psi, \alpha) \equiv P_n(\cos\psi)\,, \tag{9–24}$$

so that we are left with

$$C(\psi) = \sum_{n=2}^{\infty} c_n P_n(\cos\psi)\,. \tag{9–25}$$

The $c_n \equiv c_{n0}$ are the only coefficients that are not equal to zero. We also use the equivalent expression in terms of fully normalized harmonics:

$$C(\psi) = \sum_{n=2}^{\infty} \bar{c}_n \bar{P}_n(\cos\psi)\,. \tag{9–26}$$

The coefficients in these series, according to Sects. 1.9 and 1.10, are given by

$$\begin{aligned} c_n &= \frac{2n+1}{4\pi} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} C(\psi)\, P_n(\cos\psi) \sin\psi \; d\psi \, d\alpha \\ &= \frac{2n+1}{2} \int_{\psi=0}^{\pi} C(\psi)\, P_n(\cos\psi) \sin\psi \; d\psi \end{aligned} \tag{9–27}$$

and

$$\bar{c}_n = \frac{c_n}{\sqrt{2n+1}}\,. \tag{9–28}$$

We now determine the relation between the coefficients c_n of $C(\psi)$ in (9–25) and the coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ of Δg in (9–18). For this purpose we need an expression for $C(\psi)$ in terms of Δg, which is easily obtained by writing (9–27) more explicitly. Take the two points $P(\vartheta, \lambda)$ and $P'(\vartheta', \lambda')$ of Fig. 9.2. Their spherical distance ψ is given by

$$\cos\psi = \cos\vartheta \, \cos\vartheta' + \sin\vartheta \, \sin\vartheta' \cos(\lambda' - \lambda)\,. \tag{9–29}$$

Here ψ and the azimuth α are the polar coordinates of $P'(\vartheta', \lambda')$ with respect to the pole $P(\vartheta, \lambda)$.

The symbol M in (9–6) denotes the average over the unit sphere. Two steps are required to find it. First, we average over the spherical circle of radius ψ (denoted in Fig. 9.2 by a broken line), keeping the pole P fixed and letting P' move along the circle so that the distance PP' remains constant. This gives

$$C^* = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \Delta g(\vartheta, \lambda)\, \Delta g(\vartheta', \lambda')\; d\alpha\,, \tag{9–30}$$

where C^* still depends on the point P chosen as the pole $\psi = 0$. Second, we average C^* over the unit sphere:

$$\begin{aligned} &\frac{1}{4\pi}\int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi} C^* \sin\vartheta\, d\vartheta\, d\lambda \\ &\quad = \frac{1}{8\pi^2}\int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g(\vartheta,\lambda)\,\Delta g(\vartheta',\lambda')\sin\vartheta\, d\vartheta\, d\lambda\, d\alpha\,. \end{aligned} \tag{9–31}$$

This is equal to the covariance function $C(\psi)$, the symbol M in (9–6) now being written explicitly:

$$C(\psi) = \frac{1}{8\pi^2}\int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi}\int_{\alpha=0}^{2\pi} \Delta g(\vartheta,\lambda)\,\Delta g(\vartheta',\lambda')\sin\vartheta\, d\vartheta\, d\lambda\, d\alpha\,. \tag{9–32}$$

The coordinates ϑ', λ' in this formula are understood to be related to ϑ, λ by (9–29) with ψ = constant, but to be arbitrary otherwise; this expresses the fact that in (9–6) the average is extended over all pairs of points P and P' for which $PP' = \psi$ = constant.

To compute the coefficients c_n, substitute (9–32) into (9–27), obtaining

$$\begin{aligned} c_n &= \frac{2n+1}{2}\int_{\psi=0}^{\pi} C(\psi)\,P_n(\cos\psi)\sin\psi\, d\psi \\ &= \frac{1}{4\pi}\,\frac{2n+1}{4\pi}\int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi} \Delta g(\vartheta,\lambda)\,\Delta g(\vartheta',\lambda')\,\cdot \\ &\qquad\qquad \cdot P_n(\cos\psi)\sin\psi\, d\psi\, d\alpha \cdot \sin\vartheta\, d\vartheta\, d\lambda\,. \end{aligned} \tag{9–33}$$

Consider first the integration with respect to α and ψ. According to (1–89), we have

$$\begin{aligned} &\frac{2n+1}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi} \Delta g(\vartheta',\lambda')\,P_n(\cos\psi)\sin\psi\, d\psi\, d\alpha \\ &\quad = \frac{2n+1}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi} \Delta g(\vartheta',\lambda')\,P_n(\cos\psi)\sin\vartheta'\, d\vartheta'\, d\lambda' = \Delta g_n(\vartheta,\lambda)\,, \end{aligned} \tag{9–34}$$

the change of integration variables being evident. Hence (9–33) becomes

$$c_n = \frac{1}{4\pi}\int_{\lambda=0}^{2\pi}\int_{\vartheta=0}^{\pi} \Delta g(\vartheta,\lambda)\,\Delta g_n(\vartheta,\lambda)\sin\vartheta\, d\vartheta\, d\lambda\,. \tag{9–35}$$

This may also be written

$$c_n = M\{\Delta g\,\Delta g_n\}\,. \tag{9–36}$$

Into this we now insert (9–14), which we write

$$\Delta g(\vartheta, \lambda) = \sum_{n'=2}^{\infty} \Delta g_{n'}(\vartheta, \lambda)\,, \tag{9–37}$$

denoting the summation index by n' instead of n. We get

$$c_n = M\left\{\sum_{n'=2}^{\infty} \Delta g_{n'}\, \Delta g_n\right\} = \sum_{n'=2}^{\infty} M\{\Delta g_n\, \Delta g_{n'}\}\,. \tag{9–38}$$

According to (9–22), only the term with $n' = n$ is different from zero so that from (9–21) we finally obtain

$$c_n = M\{\Delta g_n^2\} = \sum_{m=0}^{n} (\bar{a}_{nm}^2 + \bar{b}_{nm}^2)\,. \tag{9–39}$$

Hence, c_n is the average square of the Laplace harmonic $\Delta g_n(\vartheta, \lambda)$ of degree n, or its variance. For these reasons the c_n are also called *degree variances*. The "degree covariances" are zero because of (9–22).

Equation (9–39) relates the coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ of Δg and c_n of $C(s)$ in the simplest possible way. Note that $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are coefficients of fully normalized harmonics, whereas c_n are coefficients of conventional harmonics. As a matter of fact, we may also use the a_{nm} and b_{nm} (conventional) or the $\bar{c}_n$ (fully normalized); but then (9–39) will obviously become slightly more complicated. It should be mentioned that the mathematics behind the statistical description of the gravity anomalies is the theory of *stochastic processes*. The gravity anomaly field is treated as a stationary stochastic process on a sphere; the spherical-harmonic expansions of this section are nothing but the spectral analysis of that process. A comprehensive treatment of this topic is found in Moritz (1980 a).

9.4 Interpolation and extrapolation of gravity anomalies

As pointed out in Sect. 9.1, the purpose of prediction (interpolation and extrapolation) is to supplement the gravity observations, which can be made at only relatively few points, by estimating the values of gravity or of gravity anomalies at all the other points P of the earth's surface.

If P is surrounded by gravity stations, we must interpolate; if the gravity stations are far away from P, we extrapolate. Evidently, there is no sharp distinction between these two kinds of prediction and the mathematical formulation is the same in both cases.

In order to predict a gravity anomaly at P, we must have information about the gravity anomaly function. The values observed at certain points are the most important information. In addition, we need some information on the form of the anomaly function. If the gravity measurements are very dense, then the continuity or "smoothness" of the function is sufficient – for instance, for linear interpolation. Otherwise we may try to use statistical information on the general structure of the gravity anomalies. Here we must consider two kinds of statistical correlation: the *autocorrelation* – the correlation between each other – of gravity anomalies and the *correlation* of the gravity anomalies *with height*.

Correlation with height will for the moment be disregarded; Sect. 9.7 will be devoted to this topic. The autocorrelation is characterized by the covariance function considered in Sect. 9.2.

Mathematically, the purpose of prediction is to find a function of the observed gravity anomalies Δg_1, Δg_2, ..., Δg_n in such a way that the unknown anomaly Δg_P at P is approximated by the function

$$\Delta g_P \doteq F(\Delta g_1, \Delta g_2, \ldots, \Delta g_n)\,. \tag{9–40}$$

Here Δg_i denotes the value of Δg at a point i, not a spherical harmonic! In practice, only linear functions of the Δg_i are used. If we denote the predicted value of Δg_P by $\widetilde{\Delta g}_P$, such a linear prediction has the form

$$\widetilde{\Delta g}_P = \alpha_{P1}\,\Delta g_1 + \alpha_{P2}\,\Delta g_2 + \ldots + \alpha_{Pn}\,\Delta g_n \equiv \sum_{i=1}^{n} \alpha_{Pi}\,\Delta g_i\,. \tag{9–41}$$

The coefficients α_{Pi} depend only on the relative position of P and the gravity stations 1, 2, ..., n; they are independent of the Δg_i. Depending on the way we choose these coefficients, we obtain different interpolation or extrapolation methods. Here are some examples.

Geometric interpolation

The "gravity anomaly surface", as represented by a gravity anomaly map, may be approximated by a polyhedron by dividing the area into triangles whose corners are formed by the gravity stations and passing a plane through the three corners of each triangle (Fig. 9.3). This is approximately what is done in constructing the contour lines of a gravity anomaly map by means of graphical interpolation.

Analytically, this interpolation may be formulated as follows. Let point P be situated inside a triangle with corners 1, 2, 3 (Fig. 9.3). To each point

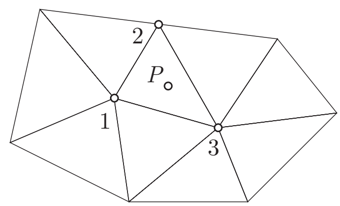

Fig. 9.3. Geometric interpolation

we assign its value Δg as its z-coordinate, so that the points 1, 2, and 3 have "spatial" coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3); x and y are ordinary plane coordinates. The plane through 1, 2, 3 has the equation

$$\begin{aligned} z = & \frac{(x_2 - x)(y_3 - y_2) - (y_2 - y)(x_3 - x_2)}{(x_2 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2)} z_1 \\ & + \frac{(x_3 - x)(y_1 - y_3) - (y_3 - y)(x_1 - x_3)}{(x_3 - x_2)(y_1 - y_3) - (y_3 - y_2)(x_1 - x_3)} z_2 \\ & + \frac{(x_1 - x)(y_2 - y_1) - (y_1 - y)(x_2 - x_1)}{(x_1 - x_3)(y_2 - y_1) - (y_1 - y_3)(x_2 - x_1)} z_3 \, . \end{aligned} \tag{9–42}$$

If we replace z_1, z_2, z_3 by Δg_1, Δg_2, Δg_3, then z is the interpolated value $\widetilde{\Delta g}_P$ at point P, which has the plane coordinates x, y. Thus,

$$\widetilde{\Delta g}_P = \alpha_{P1} \, \Delta g_1 + \alpha_{P2} \, \Delta g_2 + \alpha_{P3} \, \Delta g_3 \, , \tag{9–43}$$

where the α_{Pi} are the coefficients of z_i in the preceding equation.

Representation

Often the measured anomaly of a gravity station 1 is made to represent the whole neighborhood so that

$$\widetilde{\Delta g}_P \equiv \Delta g_1 \tag{9–44}$$

as long as P lies within a certain neighborhood of point 1. Then

$$\alpha_{P1} = 1 \, , \qquad \alpha_{P2} = \alpha_{P3} = \ldots = \alpha_{Pn} = 0 \, . \tag{9–45}$$

This method is rather crude but simple and accurate enough for many purposes.

Zero anomaly

If there are no gravity measurements in a large area – for instance, on the oceans –, then the estimate

$$\widetilde{\Delta g}_P \equiv 0 \tag{9–46}$$

is used in this area. In this trivial case all α_{Pi} are zero.

If all known gravity stations are far away, and if we know of nothing better, then this primitive extrapolation method is applied, although the accuracy is poor. At best, this method may work with isostatic anomalies.

None of these three methods gives optimum accuracy. In the next section we investigate the accuracy of the general prediction formula (9–41) and find those coefficients α_{Pi} that yield the most accurate results.

9.5 Accuracy of prediction methods

In order to compare the various possible methods of prediction, to determine their range of applicability, and to find the most accurate method, we must evaluate their accuracy.

Consider the general case of Eq. (9–41). The correct gravity anomaly at P is Δg_P, the predicted value is

$$\widetilde{\Delta g}_P = \sum_{i=1}^{n} \alpha_{Pi}\, \Delta g_i \,. \tag{9–47}$$

The difference is the error ε_P of prediction,

$$\varepsilon_P = \Delta g_P - \widetilde{\Delta g}_P = \Delta g_P - \sum_i \alpha_{Pi}\, \Delta g_i \,. \tag{9–48}$$

By squaring we find

$$\begin{aligned} \varepsilon_P^2 &= \Big(\Delta g_P - \sum_i \alpha_{Pi}\, \Delta g_i\Big)\Big(\Delta g_P - \sum_k \alpha_{Pk}\, \Delta g_k\Big) \\ &= \Delta g_P^2 - 2\sum_i \alpha_{Pi}\, \Delta g_P\, \Delta g_i + \sum_i \sum_k \alpha_{Pi}\, \alpha_{Pk}\, \Delta g_i\, \Delta g_k \,. \end{aligned} \tag{9–49}$$

Let us now form the average M of this formula over the area considered (either a limited region or the whole earth). Then we have from (9–6),

$$\begin{aligned} M\{\Delta g_i\, \Delta g_k\} &= C(i\,k) \equiv C_{ik} \,, \\ M\{\Delta g_P\, \Delta g_i\} &= C(P\,i) \equiv C_{Pi} \,, \\ M\{\Delta g_P^2\} &= C(0) \equiv C_0 \,. \end{aligned} \tag{9–50}$$

These are particular values of the covariance function $C(s)$, for $s = i\,k$, $s = Pi$, and $s = 0$; for instance, $i\,k$ is the distance between the gravity stations i and k. The abbreviated notations C_{ik} and C_{Pi} are self-explanatory.

We further set

$$M\{\varepsilon_P^2\} = m_P^2\,. \tag{9–51}$$

Thus m_P is the root mean square error of a predicted gravity anomaly at P, or briefly, the standard *error of prediction* (interpolation or extrapolation).

Taking all these relations into account, we find the average M of (9–49) to be

$$m_P^2 = C_0 - 2\sum_{i=1}^{n} \alpha_{Pi}\,C_{Pi} + \sum_{i=1}^{n}\sum_{k=1}^{n} \alpha_{Pi}\,\alpha_{Pk}\,C_{ik}\,. \tag{9–52}$$

This is the fundamental formula for the standard error of the general prediction formula (9–41). For the special cases described in the preceding section, the particular values of α_{Pi} are to be inserted.

Einstein's summation convention

At least at this point the reader will be grateful to Albert Einstein for having invented not only the theory of relativity – well, even the general theory of relativity has been used in geodesy (Moritz and Hofmann-Wellenhof 1993), but the reader of the present book will be saved from it – but also the very practical summation convention which has eradicated myriads of unnecessary summation signs from the mathematical literature. This convention simply says that, if an index occurs twice in a product, summation is automatically implied. Using this convention, the preceding equation is simply written

$$m_P^2 = C_0 - 2\ \alpha_{Pi}\,C_{Pi} + \alpha_{Pi}\,\alpha_{Pk}\,C_{ik}\,. \tag{9–53}$$

In the future we shall take this equation for granted unless stated otherwise. Such formulas are also handsome for programming (a loop).

Now back to reality in the form of examples.

As an example consider the case of representation, Eq. (9–44); all α are zero except one. Here (9–53) yields

$$m_P^2 = C_0 - 2C_{P1} + C_0 = 2C_0 - 2C_{P1}\,. \tag{9–54}$$

For the case of zero anomaly, there is $m_p^2 = C_0$, as should be expected.

Often we need not only the standard error m_P of prediction but also the correlation of the prediction errors ε_P and ε_Q at two different points P and Q, expressed by the *"error covariance"* σ_{PQ}, which is defined by

$$\sigma_{PQ} = M\{\varepsilon_P\,\varepsilon_Q\}\,. \tag{9–55}$$

If the errors ε_P and ε_Q are uncorrelated, then the error covariance $\sigma_{PQ} = 0$. From (9–48) we have generally

$$\begin{aligned}\sigma_{PQ} &= M\{(\Delta g_P - \alpha_{Pi}\,\Delta g_i)(\Delta g_Q - \alpha_{Qk}\,\Delta g_k)\}\\ &= M\{\Delta g_P\,\Delta g_Q - \alpha_{Pi}\,\Delta g_Q\,\Delta g_i - \alpha_{Qk}\,\Delta g_P\,\Delta g_k + \alpha_{Pi}\,\alpha_{Pk}\,\Delta g_i\,\Delta g_k\}\end{aligned} \tag{9–56}$$

and finally

$$\sigma_{PQ} = C_{PQ} - \alpha_{Pi}\,C_{Qi} - \alpha_{Qi}\,C_{Pi} + \alpha_{Pi}\,\alpha_{Qk}\,C_{ik}\,. \tag{9–57}$$

The notations are self-explanatory; for instance, $C_{PQ} = C(PQ)$.

The error covariance function

The values of the error covariance σ_{PQ}, for different positions of the points P and Q, form a continuous function of the coordinates of P and Q. This function is called the *error covariance function*, or briefly, the *error function*, and is denoted by $\sigma(x_P, y_P, x_Q, y_Q)$. If P and Q are different, then we simply have

$$\sigma(x_P, y_P,\, x_Q, y_Q) = \sigma_{PQ}\,; \tag{9–58}$$

if P and Q coincide, then (9–57) reduces to (9–53) so that

$$\sigma(x_P, y_P,\, x_P, y_P) = m_P^2 \tag{9–59}$$

is the square of the standard prediction error at P.

Thus the error covariances σ_{PQ} may be considered as special values of the error covariance function, just as the covariances C_{PQ} of the gravity anomalies may be considered as special values of the covariance function $C(s)$. To repeat, the error function is the covariance function of the prediction errors, defined as

$$M\{\varepsilon_P\,\varepsilon_Q\}\,, \tag{9–60}$$

whereas $C(s)$ is the covariance function of the gravity anomalies, defined as

$$M\{\Delta g_P\,\Delta g_Q\}\,. \tag{9–61}$$

The term “covariance function” in the narrower sense will be reserved for $C(s)$ – in contrast to least-squares adjustment, where “covariances” automatically mean error covariances. Covariances are “isotropic”, which means independent of directions; the error covariances are nonisotropic.

From (9–53) and (9–57) *the error function can be expressed in terms of the covariance function*; we may write more explicitly

$$\sigma(x_P, y_P,\, x_Q, y_Q) = C(P\,Q) - \alpha_{Pi}\,C(Qi) - \alpha_{Qi}\,C(Pi) + \alpha_{Pi}\,\alpha_{Qk}\,C(i\,k)\,. \tag{9–62}$$

Thus we recognize the basic role of the covariance function in accuracy studies. The error function, on the other hand, is fundamental for problems of error propagation.

9.6 Least-squares prediction

The values of α_{Pi} for the most accurate prediction method are obtained by minimizing the standard prediction error expressed by (9–53) as a function of the α. The familiar necessary conditions for a minimum are

$$\frac{\partial m_P^2}{\partial \alpha_{Pi}} \equiv -2C_{Pi} + 2\alpha_{Pk}\,C_{ik} = 0 \quad (i = 1, 2, \ldots, n) \tag{9–63}$$

or

$$C_{ik}\,\alpha_{Pk} = C_{Pi}\,. \tag{9–64}$$

This is a system of n linear equations in the n unknowns α_{Pk}; the solution is

$$\alpha_{Pk} = C_{ik}^{(-1)}\,C_{Pi}\,, \tag{9–65}$$

where $C_{ik}^{(-1)}$ denote the elements of the inverse of the symmetric matrix $[C_{ik}]$.

Substituting (9–65) into (9–41) gives

$$\widetilde{\Delta g}_P = \alpha_{Pk}\,\Delta g_k = C_{ik}^{(-1)}\,C_{Pi}\,\Delta g_k\,. \tag{9–66}$$

In matrix notation this is written

$$\widetilde{\Delta g}_P = \left[C_{P1},\, C_{P2},\, \ldots,\, C_{Pn}\right] \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}^{-1} \begin{bmatrix} \Delta g_1 \\ \Delta g_2 \\ \vdots \\ \Delta g_n \end{bmatrix}. \tag{9–67}$$

We see that *for optimal prediction we must know the statistical behavior of the gravity anomalies* through the covariance function $C(s)$.

There is a close connection between this optimal prediction method and the method of least-squares adjustment. Although they refer to somewhat different problems, both are designed to give most accurate results. The linear equations (9–64) correspond to the "normal equations" of adjustment computations. Prediction by means of formula (9–67) is therefore called *"least-squares prediction"*. A generalization to heterogeneous data is "least-squares collocation" to be treated in Chap. 10. In its most general form, least-squares collocation also includes parameter estimation by least-squares adjustment. This is an advanced subject treated in great detail in Moritz (1980 a).

It is easy to determine the accuracy of least-squares prediction. Insert the α of Eq. (9–65) into (9–53), after appropriate changes in the indices of summation. This gives

$$\begin{aligned} m_P^2 &= C_0 - 2\alpha_{Pk}\, C_{Pk} + \alpha_{Pk}\, \alpha_{Pl}\, C_{kl} \\ &= C_0 - 2C_{ik}^{(-1)} C_{Pi} C_{Pk} + C_{ik}^{(-1)} C_{Pi} C_{jl}^{(-1)} C_{Pj} C_{kl} \,. \end{aligned} \tag{9–68}$$

For the reader to appreciate the Einstein summation convention, we give this equation in its original form:

$$\begin{aligned} m_P^2 &= C_0 - 2\sum_k \alpha_{Pk}\, C_{Pk} + \sum_k \sum_l \alpha_{Pk}\, \alpha_{Pl}\, C_{kl} \\ &= C_0 - 2\sum_i \sum_k C_{ik}^{(-1)} C_{Pi} C_{Pk} + \sum_i \sum_j \sum_k \sum_l C_{ik}^{(-1)} C_{Pi} C_{jl}^{(-1)} C_{Pj} C_{kl} \,. \end{aligned} \tag{9–69}$$

But now back to normal! We have

$$C_{jl}^{(-1)}\, C_{kl} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \,. \end{cases} \tag{9–70}$$

The matrix $[\delta_{kl}]$ is the unit matrix. This formula states that the product of a matrix and its inverse is the unit matrix. Thus, we further have

$$C_{ik}^{(-1)}\, C_{jl}^{(-1)}\, C_{kl} = C_{ik}^{(-1)}\, \delta_{jk} = C_{ij}^{(-1)} \tag{9–71}$$

because a matrix remains unchanged on multiplication by the unit matrix. Hence, we get

$$\begin{aligned} m_P^2 &= C_0 - 2C_{ik}^{(-1)}\, C_{Pi}\, C_{Pk} + C_{ij}^{(-1)}\, C_{Pi}\, C_{Pj} \\ &= C_0 - 2C_{ik}^{(-1)}\, C_{Pi}\, C_{Pk} + C_{ik}^{(-1)}\, C_{Pi}\, C_{Pk} \\ &= C_0 - C_{ik}^{(-1)}\, C_{Pi}\, C_{Pk} \,. \end{aligned} \tag{9–72}$$

Thus, the standard error of least-squares prediction is given by

$$\begin{aligned} m_P^2 &= C_0 - C_{ik}^{(-1)}\, C_{Pi}\, C_{Pk} \\ &= C_0 - \left[C_{P1},\, C_{P2},\, \ldots,\, C_{Pn}\right] \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}^{-1} \begin{bmatrix} C_{P1} \\ C_{P2} \\ \vdots \\ C_{Pn} \end{bmatrix} . \end{aligned} \tag{9–73}$$

In the same way we find the error covariance in the points P and Q:

$$\sigma_{PQ} = C_{PQ} - C_{ik}^{(-1)} C_{Pi} C_{Qk}$$

$$= C_{PQ} - \left[C_{P1}, C_{P2}, \ldots, C_{Pn}\right] \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}^{-1} \begin{bmatrix} C_{Q1} \\ C_{Q2} \\ \vdots \\ C_{Qn} \end{bmatrix} . \tag{9–74}$$

These two formulas give the error covariance function for least-squares prediction. Both formulas have a form similar to that of (9–67) and are equally well suited for computations so that $\widetilde{\Delta g}$ and its accuracy can be calculated at the same time.

It is clear that, after appropriate slight changes, this theory applies automatically to *gravity disturbances* δg.

Practical considerations

Geometric interpolation (Sect. 9.4) is suited for the interpolation of point anomalies in a dense gravity net, with station distances of 10 km or less. If mean anomalies for blocks of $5' \times 5'$ or larger are needed rather than point anomalies, then some kind of representation, such as that considered in the previous section, may be simpler and hardly less accurate.

Least-squares prediction is, by its very definition, more accurate than either geometric interpolation or representation, but the improvement in accuracy is not striking. The main advantage of least-squares prediction is that it permits a systematic, purely numerical, digital processing of gravity data; gravity anomalies are stored in data bases, and gravity anomaly maps, if necessary, are generated automatically. The same formula applies to both interpolation and extrapolation so that gaps in the gravity data make no difference in the method of computation, which becomes completely schematic (Moritz 1963). For practical and computational details see Rapp (1964) and many other papers published since.

For larger station distances, of 50 km or more, prediction of individual point values becomes meaningless. In this case we must work with mean anomalies of, say, $1° \times 1°$ blocks.

9.7 Correlation with height

So far we have taken into account only the mutual correlation of the gravity anomalies, their autocorrelation, disregarding the correlation with height,

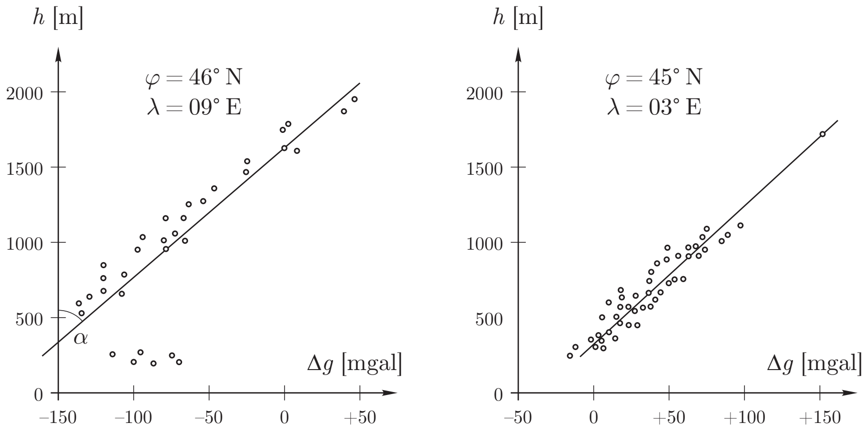

Fig. 9.4. Correlation of the free-air anomalies with height

which is important in many cases. Therefore our formulas were valid only for gravity anomalies uncorrelated with height, such as isostatic or, to a certain extent, Bouguer anomalies; or for free-air anomalies in moderately flat areas. Free-air anomalies in mountains must be treated differently.

Figure 9.4 due to U.A. Uotila shows the correlation of free-air anomalies with height. The gravity anomalies Δg are plotted against the height h. If there were an exact functional dependence between Δg and h, then all points would lie on a straight line (or, more generally, on a curve). In reality, there is only an approximate functional relation, a general trend or tendency of the free-air anomalies to increase linearly with height; exceptions, even large ones, are possible. This shows very well the meaning of correlation.

We have characterized the mutual correlation of the gravity anomalies by the "autocovariance function" (9–6),

$$C(s) = M\{\Delta g\, \Delta g'\}\,, \tag{9–75}$$

where $s = PP'$. Similarly, we may form the "cross-covariance function"

$$B(s) = M\{\Delta g\, \Delta h'\} = M\{\Delta g'\, \Delta h\}\,, \tag{9–76}$$

expressing the correlation between gravity and height, and

$$A(s) = M\{\Delta h\, \Delta h'\}\,, \tag{9–77}$$

which is the autocovariance function of the height differences

$$\Delta h = h - M\{h\}\,, \tag{9–78}$$

where the symbol $M\{h\}$ denotes the mean height of the whole area considered.

If Δg and Δh are not correlated, then the function $B(s)$ is identically zero. If this is not the case, then we should also take the height into account in our interpolation.

It is easy to extend the prediction formula (9–41) for this purpose, but this has turned out to be of little practical importance.

Application to Bouguer anomalies

Of great practical importance, however, is the question whether it is possible to render the free-air anomalies independent of height by adding a term that is proportional to the height. In other words, when is the quantity

$$z = \Delta g - b\,\Delta h\,, \tag{9–79}$$

with a certain coefficient b, uncorrelated with height? In statistical terminology, correlation with height is a *trend*, which may be capable of being removed.

The trend z has the form of a Bouguer anomaly; for a real Bouguer anomaly we have, according to Sect. 3.4,

$$b = 2\pi\,G\,\varrho\,. \tag{9–80}$$

For the density $\varrho = 2.67\ \mathrm{g/cm^3}$ we get

$$b = +0.112\ \mathrm{mgal/m}\,. \tag{9–81}$$

Let us form the covariance function $Z(s)$ between the "Bouguer anomaly" z of (9–79) and height difference Δh

$$Z(s) \equiv M\{z\,\Delta h'\} = M\{\Delta g\,\Delta h' - b\,\Delta h\,\Delta h'\} = B(s) - b\,A(s)\,. \tag{9–82}$$

If z is to be uncorrelated with h, then $Z(s)$ must be identically zero. The condition is

$$B(s) - b\,A(s) \equiv 0\,, \tag{9–83}$$

which must be satisfied for all s and a certain constant b at least approximately.

We see that the "Bouguer anomaly" z is uncorrelated with height if the functions $A(s)$ and $B(s)$ are proportional for the area considered; the constant b is then represented by

$$b = \frac{B(s)}{A(s)}\,. \tag{9–84}$$

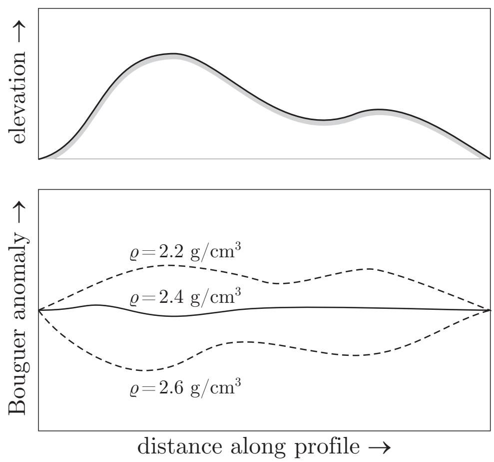

Fig. 9.5. Bouguer anomalies corresponding to different densities ϱ: the best density is $\varrho = 2.4$ g/cm^3 (no correlation); for other densities the Bouguer anomalies are correlated with height (positive correlation for $\varrho = 2.2$ g/cm^3, negative correlation for $\varrho = 2.6$ g/cm^3)

It may be shown that this is equivalent to the condition that the points of Fig. 9.4 lie approximately on a straight line. The coefficient b is then given by

$$b = \tan \alpha \tag{9–85}$$

as the inclination of the line towards the h-axis.

In practice these conditions are very often fulfilled to a good approximation. Furthermore, by computing b from Eq. (9–84) or determining it graphically by means of (9–85), we often get a value that is close to the normal Bouguer gradient (9–81).

If we assume that b depends only on the rock density ϱ, then we obtain a means for determining the average density, which is often difficult to measure directly. This is the "*Nettleton method*", used in geophysical prospecting: the coefficient b is found statistically by means of Eqs. (9–84) or (9–85), and the rock density ϱ is then computed from (9–80). Figure 9.5 illustrates the principle of this method; see also Jung (1956: p. 600).

If the condition (9–83) is fulfilled, then we may consider the "Bouguer anomaly" z as a gravity anomaly that is completely uncorrelated with height; we can directly apply to it the whole theory of the preceding sections. But even when this condition is not quite satisfied, Bouguer anomalies will in general be far less correlated with height than free-air anomalies. The fact that in (9–79) gravity is reduced to a mean height and not to sea level, is quite irrelevant in this connection because this is only a question of an additive constant. More recent developments are discussed by Moritz (1990: p. 244).

It is thus possible to consider the Bouguer reduction as a means of obtaining gravity anomalies that are less dependent on height and hence more representative than free-air anomalies. More precisely, the Bouguer anomalies take care of the dependence on the local irregularities of height. The isostatic anomalies are, in addition, also largely independent of the regional features of topography. See also Chaps. 3 and 8.