8 Modern views on the determination of the figure of the earth

8.1 Introduction

In the preceding chapters we have usually followed what might be called the conservative approach to the problems of physical geodesy using classical observations. The geodetic measurements – astronomical coordinates and azimuths, horizontal angles, gravity observations, etc. – are reduced to the geoid, and the "geodetic boundary-value problem" is solved for the geoid by means of Stokes' integral and similar formulas. The geoid then serves as a basis for establishing the position of points of the earth's surface.

The advantage of this approach is that the geoid is a level surface, capable of a simple definition in terms of the physically meaningful and geodetically important potential W. The geoid represents the most obvious mathematical formulation of a horizontal surface at sea level. This is why the use of the geoid simplifies geodetic problems and makes them accessible to geometrical intuition.

The disadvantage is that the potential W inside the earth, and hence the geoid $W =$ constant, depends on the density ϱ because of Poisson's Eq. (2–9),

$$\Delta W = -4\pi\, G\, \varrho + 2\omega^2 \,. \tag{8–1}$$

Therefore, in order to determine or to use the geoid, the density of the masses at every point between the geoid and the ground must be known, at least theoretically. This is clearly impossible, and therefore some assumptions concerning the density must be made, which is unsatisfactory theoretically, even though the practical influence of these assumptions is usually rather small.

For this reason it is of basic importance that *M.S. Molodensky* in 1945 was able to show that the physical surface of the earth can be determined from geodetic measurements alone, without using the density of the earth's crust. This requires that the concept of the geoid be abandoned. The mathematical formulation becomes more abstract and more difficult. Both the gravimetric method and the astrogeodetic method can be modified for this purpose. The gravity anomalies and the deflections of the vertical now refer to the ground, and no longer to sea level; the "height anomalies" at ground level take the place of the geoidal undulations.

These developments have considerably broadened our insight into the principles of physical geodesy and have also introduced powerful new methods for tackling classical problems. Hence their basic theoretical significance is by no means lessened by the fact that many scientists prefer to retain the geoid because of its conceptual and practical advantages.

In this chapter, we first give a concise survey of the conventional determination of the geoid by means of gravity reductions, in order to understand better the modern ideas. After an exposition of Molodensky's theory, we show how the new methods may be applied to classical problems such as gravity reduction or the determination of the geoid by gravimetric and astrogeodetic methods. It should be mentioned that the terms "modern" and "conventional" merely serve as convenient labels; they do not imply any connotation of value or preferability.

Part I: Gravimetric methods

8.2 Gravity reductions and the geoid

The integrals of Stokes and of Vening Meinesz and similar formulas presuppose that the disturbing potential T is harmonic on the geoid, which implies that there are no masses outside the geoid. This assumption – no masses outside the bounding surface – is necessary if we wish to treat any problem of physical geodesy as a boundary-value problem in the sense of potential theory. The reason is that the boundary-value problems of potential theory always involve harmonic functions, that is, solutions of Laplace's equation

$$\Delta T = 0\,. \tag{8–2}$$

This is equivalent to $\Delta V = 0$. Proof: $T = W - U$ (U is the normal potential), $\Delta W = 2\omega^2$ outside the earth (density zero, only rotation, $\Delta U = 2\omega^2$ for the same reason, hence $\Delta T = \Delta W - \Delta U = 2\omega^2 - 2\omega^2 = 0$). Since then $\Delta W = 2\omega^2$ rather than zero by Eq. (2–9), it is not quite correct to call the external gravity potential W harmonic as well, but we may nevertheless do so for simplicity. No misunderstanding is possible.

We know, for instance, that the determination of T or N from gravity anomalies Δg may be considered as a third boundary-value problem (see Sect. 1.13).

Since there are masses outside the geoid, they must be moved inside the geoid or completely removed before we can apply Stokes' integral or related

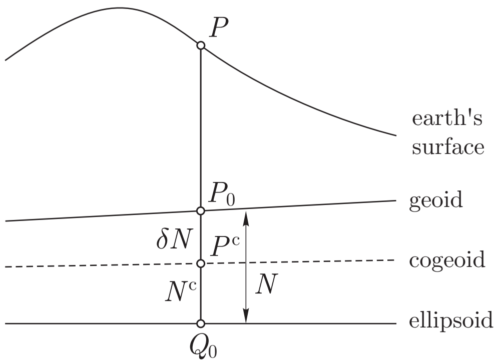

Fig. 8.1. Geoid and cogeoid

formulas. *This is the purpose of the various gravity reductions.* They were considered extensively in Chap. 3; we therefore can limit ourselves to pointing out those theoretical features that are relevant to our present problem.

If the external masses, the masses outside the geoid, are removed or moved inside the geoid, then gravity changes. Furthermore, gravity is observed at ground level but is needed at sea level. Thus, the reduction of gravity involves the consideration of these two effects, in order to obtain boundary values on the geoid.

This *regularization* of the geoid by removing the external masses unfortunately also changes the level surfaces and hence, in general, the geoid. This is the *indirect effect*; the changed geoid is called the *cogeoid* or the regularized geoid.

The principle of this method may be described as follows (Jung 1956: p. 578); see Fig. 8.1.

1. The masses outside the geoid are, by computation, either removed entirely or else moved inside the geoid. The effect of this procedure on the value of gravity g at the station P is considered.

2. The gravity station is moved from P down to the geoid, to the point P_0. Again, the corresponding effect on the gravity is considered.

3. The indirect effect, the distance $\delta N = P_0 P^c$, is obtained by dividing the change in potential at the geoid, δW, by normal gravity (Bruns' theorem):
$$\delta N = \frac{\delta W}{\gamma}\,. \tag{8–3}$$

4. The gravity station is now moved from the geoidal point P_0 to the

cogeoid, to the point P^c (hence the notation with upper index c). This gives the boundary value of gravity at the cogeoid, g^c.

5. The shape of the cogeoid is computed from the reduced gravity anomalies

$$\Delta g^c = g^c - \gamma \tag{8–4}$$

by Stokes' formula, which gives $N^c = QP^c$.

6. Finally, the geoid is determined by considering the indirect effect. The geoidal undulation N is thus obtained as

$$N = N^c + \delta N \,. \tag{8–5}$$

Remark. At first sight it may seem that the masses between the geoid and the cogeoid should be removed if the cogeoid happens to be below the geoid, because Stokes' formula is applied to the cogeoid. However, this is not necessary, and therefore we need not be concerned with a "secondary indirect effect". The argument is a little too technical to be presented here; see Moritz (1965: p. 26).

In principle, every gravity reduction that gives boundary values at the geoid is equally suited for the determination of the geoid, provided the indirect effect is properly taken into account. Thus, the selection of a good reduction method should be made from other points of view, such as the geophysical meaning of the reduced gravity anomalies, the simplicity of computation, the feasibility of interpolation between the gravity stations, the smallness or even absence of the indirect effect, etc. (see Sect. 3.7).

The *Bouguer reduction* corresponds to a complete removal of the external masses. In the *isostatic reduction*, these masses are shifted vertically downward according to some theory of isostasy. In Helmert's *condensation reduction*, the external masses are compressed to form a surface layer on the geoid. The Bouguer reduction and especially the isostatic reduction (in modern terminology *topographic-isostatic reduction*) are used as auxiliary quantities for computational purposes, especially to facilitate interpolation.

The *free-air anomaly* is nowadays used in three senses:

1. at ground level (on the physical surface of the earth) it is simply the gravity anomaly in the sense of Molodensky (Sect. 8.4);
2. at sea level it may be identified with the analytical continuation of the Molodensky anomaly from ground down to sea level. This will be considered in detail in Sect. 8.6. A final review will be found in Sect. 8.15.

3. The free-air anomaly can be theoretically interpreted as an approximation of the classical condensation anomaly in the sense of Helmert (Sect. 3.9). This is one of the interpretations of the frequent practice to simply apply Stokes' formula to the classical free-air anomaly, where only the standard normal free-air reduction is applied to measured gravity g, see Eq. (8–6) below.

   This is pretty rigorously the gravity anomaly in the sense of Molodensky (item 1 above), so there is another interpretation of this frequent practice: it is a (conscious or unconscious) use of Molodensky's method in the zero approximation (i.e., only Stokes' formula without Molodensky correction g_1, see Sect. 8.6). Of course, this works only in a reasonably flat terrain.

   Important remark. Curiously enough, it helps if the terrain correction (Sect. 3.4) is applied; this is explained in Moritz (1980 a: Sect. 48) as some kind of Molodensky correction g_1 and in Moritz (1990: p. 244) by isostatic reduction.

Also the *Poincaré–Prey reduction* is quite different (Sect. 3.5). It gives the actual gravity inside the earth. It does not give boundary values but is used for orthometric heights (Chap. 4).

In all reduction methods it is necessary to know the density of the masses above the geoid. In practice, this involves some kind of an assumption – for instance, putting $\varrho = 2.67\ \mathrm{g\,cm^{-3}}$. A second assumption is usually made in the free-air reduction, which is part of the reduction of gravity to the geoid: the actual free-air gravity gradient is assumed to be equal to the normal gradient

$$\frac{\partial \gamma}{\partial h} \doteq -0.3086\ \mathrm{mgal\,m^{-1}}\,. \tag{8–6}$$

These two assumptions falsify our results, at least theoretically.

The second assumption can be avoided by using the actual free-air gradient as computed by the methods of Sect. 2.20. The anomalies Δg to be used in formula (2–394) must be gravity anomalies reduced to the geoid: gravity g after steps 1 and 2 of the above description, minus normal gravity γ on the ellipsoid. This presupposes that in step 2 a preliminary free-air reduction using the normal gradient has been applied first.

Deflections of the vertical

The indirect effect affects the deflection of the vertical as well as the geoidal height. We have found

$$N = N^{\mathrm{c}} + \delta N\,, \tag{8–7}$$

where N^{c} is the undulation of the cogeoid, the immediate result of Stokes' formula, and δN is the indirect effect. By differentiating N in a horizontal direction, we get the deflection component along this direction:

$$\varepsilon = -\frac{\partial N}{\partial s} = -\frac{\partial N^{\mathrm{c}}}{\partial s} - \frac{\partial(\delta N)}{\partial s}\,. \tag{8–8}$$

This means that we must add to the immediate result of Vening Meinesz' formula, $-\partial N^{\mathrm{c}}/\partial s$, a term representing the horizontal derivative of δN (see also Sect. 3.7).

To repeat, the main purpose is to obtain a simple boundary surface. The geoid approximated by an ellipsoid or even a sphere is a much easier boundary surface than the physical surface of the earth, to which we turn now.

8.3 Geodetic boundary-value problems

It is, however, quite easy to understand the general principles. In space we have the well-known fact that the gravity vector $\mathbf{g}$ and the gravity potential (geopotential) W are related by

$$\mathbf{g} = \operatorname{grad} W \equiv \left[\frac{\partial W}{\partial x}\,,\ \frac{\partial W}{\partial y}\,,\ \frac{\partial W}{\partial z}\right], \tag{8–9}$$

which shows that the force $\mathbf{g}$ is the gradient vector of the potential.

Let S be the earth's topographic surface and let W and $\mathbf{g}$ be the geopotential and the gravity vector on this surface. Then there exists a relation

$$\mathbf{g} = f(S, W)\,, \tag{8–10}$$

the gravity vector $\mathbf{g}$ on S is a function of the surface S and the geopotential W on it. This can be seen in the following way. Let the surface S and the geopotential W on S be given. The gravitational potential V is obtained by subtracting the potential of the centrifugal force Φ, which is simple and perfectly known (Sect. 2.1):

$$V = W - \Phi\,. \tag{8–11}$$

The potential V outside the earth is a solution of Laplace's equation $\Delta V = 0$ and consequently *harmonic* (Sect. 1.3). Thus, knowing V *on* S, we can obtain V outside S by solving Dirichlet's boundary-value problem, the first boundary-value problem of potential theory, which is practically always uniquely solvable (Sect. 1.12) at least if V is sufficiently smooth on S. After having found V as a function in space outside S, we obtain the gravitational force grad V. Adding the well-known and simple vector of the centrifugal force, we obtain the gravity vector $\mathbf{g}$ outside and, by continuity, on S.

This is precisely what (8–10) means. The modern general concept of a *function* can be explained as a *rule of computation*, indicating that given S and W on S, we can uniquely calculate $\mathbf{g}$ on S. Note that f is not a function in the elementary sense but rather a "nonlinear operator", but we disregard this for the moment. Therefore we may formulate:

(1) *Molodensky's boundary-value problem* is the task to determine S, the earth's surface, if $\mathbf{g}$ and W on it are given. Formally, we have to solve (8–10) for S:

$$S = F_1(\mathbf{g}, W)\,, \tag{8–12}$$

that is, we get *geometry from gravity.*

(2) *GPS boundary-value problem.* Since we have GPS at our disposal, we can consider S as known, or at least determinable by GPS. In this case, the geometry S is known, and we can solve (8–10) for W:

$$W = F_2(S, \mathbf{g})\,, \tag{8–13}$$

that is, we get *potential from gravity.* As we shall see, this is far from being trivial: we have now a method to *replace leveling*, a tedious and time-consuming old-fashioned method, *by GPS leveling*, a fast and modern technique (Sect. 4.6).

In spite of all similarities, we should bear in mind a fundamental difference: (8–13) solves a *fixed-boundary problem* (boundary S given), whereas (8–12) solves a *free-boundary problem*: the boundary S is a priori unknown ("free"). Fixed-boundary problems are usually simpler than free ones.

This is only the principle of both solutions. The formulation is quite easy to understand. The direct implementation of these formulas is difficult, however, because that would imply the solution of "hard inverse function theorems" of nonlinear functional analysis. For numerical computations, we know series solutions, in the form of "Molodensky series", which are sufficient for all present purposes and which can, furthermore, be derived in an elementary fashion, without needing integral equations (Molodenski 1958; Molodenskii et al. 1962; Moritz 1980 a: Sect. 45). Here we shall outline the known elementary solution for Molodensky's problem and immediately extend it to the GPS problem. Both problems will be solved by very similar Molodensky series.

The simplest possible example

Let the boundary surface S be a sphere of radius R. The earth is represented by this sphere which is considered homogeneous and nonrotating. The potential W is identical to the gravitational potential V, so that on the surface S we have constant values

$$\begin{aligned} W &= \frac{GM}{R}\,, \\ g &= \frac{GM}{R^2}\,. \end{aligned} \tag{8–14}$$

Knowing W and g, we have

$$R = \frac{W}{g}\,, \tag{8–15}$$

the radius of the sphere S. Thus, we have solved Molodensky's problem in this trivial but instructive example. We have indeed got geometry (i.e., R) from physics (i.e., g and W)!

8.4 Molodensky's approach and linearization

We have just seen that the reduction of gravity to sea level necessarily involves assumptions concerning the density of the masses above the geoid. This is equally true of other geodetic computations when performed in the conventional way.

To see this, consider the problem of computing the ellipsoidal coordinates φ, λ, h from the natural coordinates Φ, Λ, H, as described in Chap. 5. The geometric ellipsoidal height h above the ellipsoid is obtained from the orthometric height H above the geoid and the geoidal undulation N by

$$h = H + N\,. \tag{8–16}$$

The determination of N was considered in Chap. 2 and elsewhere in this book. To compute H from the results of leveling, we need the mean gravity $\bar{g}$ along the plumb line between the geoid and the ground (Sect. 4.3). Since gravity g cannot be measured inside the earth, we compute it by Prey's reduction, for which we must know the density of the masses above the geoid.

The ellipsoidal coordinates φ and λ are obtained from the astronomical coordinates Φ and Λ and the deflection components ξ and η by

$$\varphi = \Phi - \xi\,, \qquad \lambda = \Lambda - \eta \sec\varphi\,. \tag{8–17}$$

The coordinates Φ and Λ are measured on the ground; ξ and η can be computed for the geoid by Vening Meinesz' formula, the indirect effect being taken into account according to Sect. 8.2. To apply the above formulas, either Φ and Λ must be reduced down to the geoid or ξ and η must be reduced up to the ground. In both cases this involves the reduction for the curvature of the plumb line (Sect. 5.15), which also depends on the mean value $\bar{g}$ through its horizontal derivatives. Hence Prey's reduction enters here too.

Thus we see that in the conventional approach to the problems of physical geodesy we must know the density of the outer masses or make assumptions concerning it. To avoid this, Molodensky proposed a different approach in 1945.

Figure 8.2 shows the geometrical principles of this method, which is essentially a linearization of Eq. (8–10). The ground point P (i.e., point on the earth's surface S) is again projected onto the ellipsoid according to Helmert. However, the ellipsoidal height h is now determined by

$$h = H^* + \zeta \,, \tag{8–18}$$

the *normal height* H^* replacing the orthometric height H, and the *height anomaly* ζ replacing the geoidal undulation N.

This will be clear if one considers the surface whose normal potential U at every point Q is equal to the actual potential W at the corresponding point P, so that $U_Q = W_P$, corresponding points P and Q being situated on the same ellipsoidal normal. This surface is called the *telluroid* (Hirvonen 1960, 1961). The vertical distance from the ellipsoid to the telluroid is the normal height H^* (Sect. 4.4), whereas the ellipsoidal height h is the vertical distance from the ellipsoid to the earth's surface. Thus, the difference between these two heights is the height anomaly

$$\zeta = h - H^* \,, \tag{8–19}$$

closely corresponding to the geoidal undulation $N = h - H$, which is the difference between the ellipsoidal and the orthometric height.

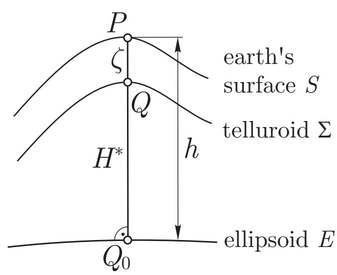

Fig. 8.2. Telluroid, normal height H^*, and height anomaly ζ

The normal height H^*, and hence the telluroid Σ, can be determined by leveling combined with gravity measurements, according to Sect. 4.4. First the geopotential number of P, $C = W_0 - W_P$, is computed by

$$C = \int_0^P g\,dn\,, \tag{8–20}$$

where g is the measured gravity and dn is the leveling increment. The normal height H^* is then related to C by an *analytical* expression such as (4–63),

$$H^* = \frac{C}{\gamma_{Q_0}}\left[1 + (1 + f + m - 2f\sin^2\varphi)\frac{C}{a\,\gamma_{Q_0}} + \left(\frac{C}{a\,\gamma_{Q_0}}\right)^2\right], \tag{8–21}$$

where γ_{Q_0} is the normal gravity at the ellipsoidal point Q_0. Note that H^* is independent of the density.

The normal height H^* of a ground point P is identical with the ellipsoidal height h, the height above the ellipsoid, of the corresponding telluroid point Q. If the geopotential function W were equal to the normal potential function U at every point, then Q would coincide with P, the telluroid would coincide with the physical surface of the earth, and the normal height of every point would be equal to its ellipsoidal height. Actually, however, $W_P \neq U_P$; hence the difference

$$\zeta_P = h_P - H^*_P = h_P - h_Q \tag{8–22}$$

is not zero. This explains the term "height anomaly" for ζ.

The gravity anomaly is now defined as

$$\Delta g = g_P - \gamma_Q\,; \tag{8–23}$$

it is the difference between the actual gravity as measured on the ground and the normal gravity on the telluroid. The normal gravity on the telluroid, which we shall briefly denote by γ, is computed from the normal gravity at the ellipsoid, γ_{Q_0}, by the normal free-air reduction, but now applied *upward*:

$$\gamma \equiv \gamma_Q = \gamma_{Q_0} + \frac{\partial\gamma}{\partial h}H^* + \frac{1}{2!}\frac{\partial^2\gamma}{\partial h^2}H^{*2} + \cdots\,. \tag{8–24}$$

For this reason, the new gravity anomalies (8–23) are called *free-air anomalies*. They are *referred to ground level*, whereas the conventional gravity anomalies have been referred to sea level. Therefore, the new free-air anomalies have nothing in common with a free-air reduction of actual gravity to sea level, except the name. This distinction should be carefully kept in mind.

A direct formula for computing γ at Q is (2–215),

$$\gamma = \gamma_{Q_0}\left[1 - 2(1 + f + m - 2f\sin^2\varphi)\frac{H^*}{a} + 3\left(\frac{H^*}{a}\right)^2\right], \tag{8–25}$$

where γ_{Q_0} is the corresponding value on the ellipsoid.

The height anomaly ζ may be considered as the distance between the geopotential surface $W = W_P =$ constant and the corresponding spheropotential surface $U = W_P =$ constant at the point P. In Sect. 2.14 (Fig. 2.15), we have denoted this distance by N_P and have found that Bruns' formula (2–237) also applies to this quantity. Hence, for $\zeta = N_P$ we have

$$\zeta = \frac{T}{\gamma}\,, \tag{8–26}$$

where $T = W_P - U_P$ is the disturbing potential at ground level, and γ the normal gravity at the telluroid.

It may be expected that ζ is connected with the ground-level anomalies Δg by an expression analogous to Stokes' formula for the geoidal height N. This is indeed true. However, the telluroid is not a level surface, and to every point P on the earth's surface corresponds in general a different geopotential surface $W = W_P$. Therefore, the relation between Δg and ζ in the new theory is considerably more complicated than for the geoid. In Molodensky's original formulation, the problem involves an integral equation, which may be solved by an iteration, the first term of which is given by Stokes' formula. We shall use an equivalent but much simpler approach without integral equation.

Finally, we remark that we may also plot the height anomalies ζ above the ellipsoid. In this way we get a surface that is identical with the geoid over the oceans, because there $\zeta = N$, and is very close to the geoid anywhere else. This surface has been called the *quasigeoid* by Molodensky. However, the quasigeoid is not a level surface and has no physical meaning whatever. It must be considered as a concession to conventional conceptions that call for a geoidlike surface. From this point of view, the normal height of a point is its elevation above the quasigeoid, just as the orthometric height is its elevation above the geoid.

Gravity disturbance

As usual, the gravity disturbance is defined by

$$\delta g = g_P - \gamma_P\,. \tag{8–27}$$

It is a typical new feature introduced into the practice of physial geodesy by GPS, because GPS determines the ellipsoidal coordinates φ, λ, h directly at the surface point P, so that now δg can be considered observational data instead of Δg.

Linearization

The linearization applies equally well for the Molodensky problem and the GPS problem. The geometry is familiar (Fig. 8.2).

We recall the surface Σ, the *telluroid*, which is defined by the condition

$$U(Q) = W(P) \,. \tag{8–28}$$

We note that (8–28) is the surface equivalent to the classical relation for sea level (Fig. 8.3)

$$U(Q_0) = W(P_0) \,. \tag{8–29}$$

Equation (8–28) would apply with

$$W(P_0) = W_0 = \text{ constant} \tag{8–30}$$

if S were an equipotential surface, the geoid, which is the case only over the oceans with the usual simplifying assumption that the surface of the ocean is an equipotential surface not changing with time (Fig. 8.3).

Molodensky's theory does not use the geoid directly but the physical earth's surface. We repeat once more that this is Molodensky's epochal idea which radically changed the course of physical geodesy since 1945.

We shall, however, use the fictitious case of S being an equipotential surface, but only as a first (or zero-order) assumption in a perturbation approach for the real earth's surface (Molodensky series). This first approximation is the spherical case to be considered in the next section.

Now we consider the linearization in more detail. The ellipsoidal height h is directly determined by GPS. It may be decomposed into

$$h = H^* + \zeta \,. \tag{8–31}$$

Here, H^* is the normal height and ζ is the height anomaly, whose definitions are seen from Fig. 8.2. In the GPS case we do know the earth's surface S directly, but the telluroid Σ and the height anomalies ζ are still required for formulating the boundary condition, just as the knowledge of the geoid does not make superfluous the reference ellipsoid.

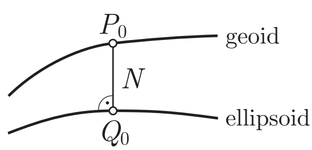

Fig. 8.3. Geoid and ellipsoid

The definition of the *gravity anomaly* Δg and the *gravity disturbance* δg has, on the earth's surface, the same form as in the classical case of geoid and sea level:

$$\Delta g = g_P - \gamma_Q = -\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T \,, \tag{8–32}$$

$$\delta g = g_P - \gamma_P = -\frac{\partial T}{\partial h} \,. \tag{8–33}$$

The gravity disturbance δg has become practically important only through GPS, since h, the ellipsoidal height of P, can be measured using GPS and hence γ_P, the normal gravity γ at P, can be determined.

As usual, Bruns' formula applies at P_0 (classical geoid height N) and P (Molodensky height anomaly ζ) as well:

$$N = \frac{T(P_0)}{\gamma} \,, \tag{8–34}$$

$$\zeta = \frac{T(P)}{\gamma} \,, \tag{8–35}$$

with some approximate value for γ such as γ_{45°. Equation (8–32) can be reformulated as the *boundary conditions for the Molodensky problem*

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T + \Delta g = 0 \,, \tag{8–36}$$

cf. (2–251), and *for the GPS problem*, cf. (2–252),

$$\frac{\partial T}{\partial h} + \delta g = 0 \,. \tag{8–37}$$

These two boundary conditions apply at the surface S (Molodensky) and at sea level as well.

Finally we introduce the spherical approximation, disregarding the flattening f in the equations (which are linear relations between small quantities).

Note: The spherical approximation is a formal operation (disregarding f in small *ellipsoidal* quantities) and does not mean using a "reference sphere" instead of a reference ellipsoid in any geometrical sense (Moritz 1980 a: p. 15). This would imply geoidal heights on the order of 20 km!

Then (8–36) and (8–37) reduce to

$$\frac{\partial T}{\partial r} + \frac{2}{r} T + \Delta g = 0 \,, \tag{8–38}$$

$$\frac{\partial T}{\partial r} + \delta g = 0 \,. \tag{8–39}$$

These equations, for the Molodensky and the GPS problem, are valid both at sea level (classical) and at S (Molodensky).

8.5 The spherical case

As we have agreed, we work formally with a sphere (the reference ellipsoid stays at its geometric place!). This means putting $r = R =$ constant. Furthermore, we assume (fictitiously!) that S is a level surface.

Expanding T and Δg into a series of Laplace spherical harmonics, see (2–322) and (2–320), we find

$$T(\vartheta, \lambda) = \sum_{2}^{\infty} T_n(\vartheta, \lambda)\,, \tag{8–40}$$

$$\Delta g(\vartheta, \lambda) = \sum_{2}^{\infty} \Delta g_n(\vartheta, \lambda) \tag{8–41}$$

on the surface of the sphere, whence by (8–38) and (2–321) with $r = R$,

$$T = R \sum_{n=2}^{\infty} \frac{\Delta g_n}{n-1}\,. \tag{8–42}$$

The summation starts conventionally with $n = 2$, rather than $n = 0$, for several reasons, one of them being that $n = 1$ would lead to a zero denominator in (8–42).

Using (2–325) and (2–326) leads to the well-known Stokes' formula

$$T = \frac{R}{4\pi} \iint_{\sigma} S(\psi)\, \Delta g \, d\sigma\,, \tag{8–43}$$

where

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\psi)\,, \tag{8–44}$$

where $P(\cos\psi)$ are Legendre polynomials. Here ψ denotes the spherical distance from the point at which T is to be computed.

In exactly the same way, we obtain for the gravity disturbance with the boundary condition (8–39), summarizing the derivation in Sect. 2.18,

$$\delta g(\vartheta, \lambda) = \sum_{0}^{\infty} \delta g_n(\vartheta, \lambda)\,, \tag{8–45}$$

$$T(\vartheta, \lambda) = R \sum_{n=0}^{\infty} \frac{\delta g_n}{n+1}\,, \tag{8–46}$$

and the formula of Neumann–Koch

$$T = \frac{R}{4\pi} \iint_\sigma K(\psi)\, \delta g \, d\sigma \,, \tag{8–47}$$

where

$$K(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi) \tag{8–48}$$

and, by summation of this series,

$$K(\psi) = \frac{1}{\sin(\psi/2)} - \ln\left(1 + \frac{1}{\sin(\psi/2)}\right) \tag{8–49}$$

being the Neumann–Koch function.

So in the GPS boundary problem on the sphere, the solution (8–47) is completely analogous to the formula of Stokes (8–43) for the classical problem.

The fact that the GPS problem is conceptually simpler (fixed-boundary surface) than Molodensky's problem (free-boundary surface) is expressed by the fact that Stokes' function must start with $n = 2$, since $n = 1$ gives a zero denominator, whereas Neumann–Koch's function (8–48) is regular for all n.

In both cases, the height anomaly ζ (here the geoidal height) is given by Bruns' formula

$$\zeta = \frac{T}{\gamma} \doteq \frac{T}{\gamma_0} \,. \tag{8–50}$$

In the spherical approximation, γ may be, in formulas of Bruns' and Stokes' type, replaced by our usual mean value $\gamma_0 = \gamma_{45°}$.

We will see that these spherical solutions form the base for an elementary solution of Molodensky's problem and the GPS problem for the earth's surface. We only mention the well-known fact that, for the earth's surface S, these two problems are *oblique-derivative problems*, since the direction of the plumb line does not coincide with the normal to the earth's surface, at least on land. Thus the GPS boundary problem *for S* is not a spherical Neumann problem, which always involves the normal derivative!

8.6 Solution by analytical continuation

8.6.1 The idea

The idea is very simple (Fig. 8.4). Our observations Δg or δg, given on the earth's surface S, are "reduced", or rather "analytically continued" (upward or downward, see below and Fig. 8.5), to a level surface (or normal level

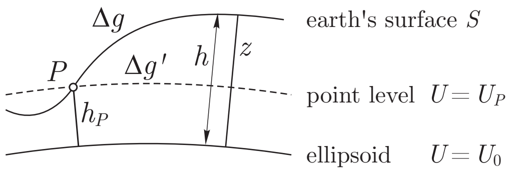

Fig. 8.4. Analytical continuation from the earth's surface to point level

surface $U = U_P$, which for our purpose is the same). In the spherical approximation, both surfaces $U = U_P$ and $U = U_0$ are concentric spheres, but only in the precise sense of the spherical approximation as explained above.

We also use the term "harmonic continuation" because the analytically continued function satisfies Laplace's equation. This will be explained in detail later.

An expansion into a Taylor series gives immediately

$$\begin{aligned}\Delta g &= \Delta g^* + z\,\frac{\partial \Delta g^*}{\partial z} + \frac{1}{2!}\,z^2\,\frac{\partial^2 \Delta g^*}{\partial z^2} + \frac{1}{3!}\,z^3\,\frac{\partial^3 \Delta g^*}{\partial z^3} + \cdots \\ &= \Delta g^* + \sum_{n=1}^{\infty} \frac{1}{n!}\,z^n\,\frac{\partial^n \Delta g^*}{\partial z^n}\,,\end{aligned} \tag{8–51}$$

where

$$z = h - h_P \tag{8–52}$$

is the elevation difference with respect to the computation point P. For the present, we assume the series (8–51) to be convergent. Here Δg^* is the gravity anomaly at point level (Fig. 8.4). The use of a Taylor series is typical for *analytical continuation*. For instance, Taylor series are a standard tool for analytical continuation of functions of a complex variable.

8.6.2 First-order solution

It is particularly easy to give a solution as a first approximation. With γ_0 from (8–50) we have

$$\zeta = \frac{R}{4\pi\,\gamma_0} \iint_\sigma \left(\Delta g - \frac{\partial \Delta g}{\partial h}\,h\right) S(\psi)\,d\sigma + \frac{\partial \zeta}{\partial h}\,h\,. \tag{8–53}$$

This follows from the geometrical interpretation of this equation which is evident from Fig. 8.5 a. We see that the free-air anomalies Δg at ground level are "reduced" downward to sea level to become

$$\Delta g^{\text{harmonic}} = \Delta g - \frac{\partial \Delta g}{\partial h}\,h \tag{8–54}$$

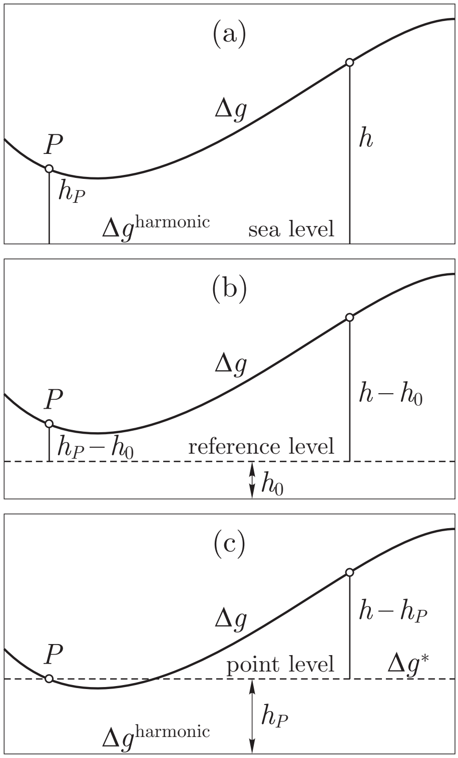

Fig. 8.5. Harmonic continuation to sea level (a), to an arbitrary level (b), and to the level of point P (c)

(the superscript "harmonic" denotes harmonic continuation to sea level; see Fig. 8.5 and the paragraph "A note on terminology" below); then Stokes' integral gives height anomalies at sea level which are reduced upward to ground level by adding the term $\frac{\partial \zeta}{\partial h} h$.

Harmonic continuation to point level

The elevation h in (8–53) is taken above sea level (see Fig. 8.5 a). If we examine the arguments leading to this equation, we will find that the sea level is not distinguished from any other level. If we reckon the elevation above some other reference level, which has the elevation h_0 above sea level, we must replace h by $h - h_0$ (see Fig. 8.5 b). Thus (8–53) is equivalent to

$$\zeta = \frac{R}{4\pi\,\gamma_0} \iint\limits_{\sigma} \left[\Delta g - \frac{\partial \Delta g}{\partial h} (h - h_0) \right] S(\psi)\, d\sigma + \frac{\partial \zeta}{\partial h} (h - h_0)\,. \qquad (8\text{–}55)$$

In particular we may take as reference level the level of the point P itself, so that

$$h_0 = h_P \,, \tag{8–56}$$

where P is the point at which the height anomaly ζ is computed. If this choice is made, the last term in the above expression will be zero, because outside the integral h always means h_P, so that $h - h_0 = h_P - h_P = 0$. Thus we have

$$\zeta = \frac{R}{4\pi\,\gamma_0} \iint\limits_{\sigma} \left[\Delta g - \frac{\partial \Delta g}{\partial h}\,(h - h_P)\right] S(\psi)\, d\sigma \,. \tag{8–57}$$

This formula is particularly simple; geometrically it means that the free-air anomalies are "reduced" (in the sense of "analytically or harmonically continued") from the ground to the level of the computation point P (see Fig. 8.5 b). Thus, the reference level is different for different computation points.

As we have already indicated at the beginning of Sect. 8.6.1, Fig. 8.5 c shows that harmonic continuation by Eq. (8–57) is *upward* for surface points *below* the level of P and *downward* for surface points *above* the level of P.

Important remark

Equation (8–57) is really a genuine spherical Stokes formula applied to a "reference sphere", namely, to the spherical "point level"! An immediate consequence: this formula can be simply differentiated *horizontally* to give a genuine Vening Meinesz formula in the sense of Sect. 2.19 for the *deflections of the vertical*. This remark is relevant for Sect. 8.7.

Vertical derivative

The vertical derivative $\partial/\partial r$ can be expressed in terms of surface values by the well-known spherical formula (Sect. 1.14)

$$\frac{\partial f}{\partial r} = -\frac{1}{R}\, f + \frac{R^2}{2\pi} \iint\limits_{\sigma} \frac{f - f_Q}{l_0^3}\, d\sigma \,. \tag{8–58}$$

Q is the surface point where $\partial f/\partial r$ is computed and to which f in the first term on the right-hand side refers, σ denotes the unit sphere, and

$$l_0 = 2R \sin\frac{\psi}{2} \,. \tag{8–59}$$

This gives $\partial \Delta g/\partial r$ if we put $f = \Delta g$ in (8–58). We may also introduce the linear gradient operator L by

$$L(f) = \frac{R^2}{2\pi} \iint\limits_{\sigma} \frac{f - f_Q}{l_0^3}\, d\sigma \,. \tag{8–60}$$

(The first term on the right-hand side of (8–58) is much smaller and can be omitted.)

The term $\partial\zeta/\partial r$ no longer occurs in (8–57) as it did in (8–53) and (8–55), and will not be needed.

Computational formulas; the Molodensky correction

Our computational formula is (8–57). We split it up as follows: The free-air anomaly Δg is continued (downward or upward) from ground level to the level of point P, obtaining

$$\Delta g^* = \Delta g + g_1 \,, \tag{8–61}$$

where the Molodensky correction is

$$g_1 = -\frac{\partial \Delta g}{\partial h}(h - h_P) = -\frac{\partial \Delta g}{\partial r}\,(h - h_P) \tag{8–62}$$

(in spherical approximation) with

$$\frac{\partial \Delta g}{\partial r} = \frac{R^2}{2\pi} \iint\limits_{\sigma} \frac{\Delta g - \Delta g_Q}{l_0^3}\, d\sigma \,. \tag{8–63}$$

Then we finally get

$$\zeta = \zeta_0 + \zeta_1 \,, \tag{8–64}$$

where

$$\zeta_0 = \frac{R}{4\pi\,\gamma_0} \iint\limits_{\sigma} \Delta g\, S(\psi)\, d\sigma \tag{8–65}$$

is the simple Stokes formula applied to ground-level free-air anomalies Δg, and the Molodenski correction for ζ is

$$\zeta_1 = \frac{R}{4\pi\,\gamma_0} \iint\limits_{\sigma} g_1\, S(\psi)\, d\sigma \,. \tag{8–66}$$

This is the first-order solution, or linear solution.

Important remark

Please note carefully that we are using “linear”, or “first-order”, in two very different senses:

- *general linearization*, linear in quantities of the anomalous potential, such as N or ζ, as introduced in Sect. 2.12 and Sect. 8.4 and implied everywhere throughout the book, and

- *linear approximation in h* used very generally in first-order "Molodensky corrections" such as (8–62) or (8–66) but not in (8–67) or (8–68).

In fact, to a higher approximation

$$\Delta g^* = \Delta g + g_1 + g_2 + g_3 + \cdots \tag{8–67}$$

and

$$\zeta = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \cdots . \tag{8–68}$$

Generalizing (8–66), we have

$$\zeta_i = \frac{R}{4\pi\,\gamma_0} \iint_\sigma g_i\, S(\psi)\, d\sigma\,, \tag{8–69}$$

where $i = 1, 2, 3, \ldots$. For the deflection of the vertical we have similar expressions, see Sect. 8.7; compare also (8–75) and (8–76).

8.6.3 Higher-order solution

The following recursion formulas are somewhat advanced and may be omitted. From Moritz (1980 a: Sect. 45) we may take the recursion formula for the correction terms g_n, which are evaluated recursively by

$$g_n = -\sum_{r=1}^{n} z^r\, L_r(g_{n-r})\,, \tag{8–70}$$

starting from

$$g_0 = \Delta g\,. \tag{8–71}$$

Here the operator L_n is also defined recursively:

$$L_n(\Delta g) = n^{-1} L_1[L_{n-1}(\Delta g)] \tag{8–72}$$

starting with

$$L_1 = L \tag{8–73}$$

with the gradient operator L defined above, (8–60), and z given by (8–52).

8.6.4 Problems of analytical continuation

Analytical continuation comes from the theory of complex variables and means extending the domain, on which the function is defined, by the use of Taylor series. Complex functions always satisfy Laplace's equations in two dimensions and are therefore harmonic.

Also in three dimensions, functions satisfying Laplace's equation are called harmonic, as we know well. Analytical continuation is again best defined by Taylor series, and analytical continuation is frequently called *harmonic continuation* (Kellogg 1929: Chap. X).

Above we have been misusing the all-round word "reduced" in the sense of "analytical" or "harmonic" continuation and will continue to do so for brevity. As we have seen in Sect. 8.2 and will see in Sect. 8.9, it is not a gravity reduction in the standard sense of explicit mass removal. The Taylor series whose first term is (8–54) is an *analytical* operation performed on the external potential directly at ground level, preserving the Laplace equation $\Delta W = 0$. (In fact, $\Delta W = 2\omega^2$, but let us, as we did in (8–2), for a while forget earth rotation, which implies $\omega = 0$ and $W = V$.) Thus, it is a harmonic function and our "reduction" is really *analytical continuation* as a harmonic function or briefly *harmonic continuation.* **Harmonic continuation is the key notion in modern physical geodesy, from Molodensky's problem to least-squares collocation.** Its full meaning will gradually emerge in what follows, as a notion which is surprisingly simple and general. Symbols like $\Delta g^{\text{harmonic}}$ will relate to harmonic continuation. In what follows, we shall sometimes continue to use "reduce downward" or "continue downward" instead of "harmonically continue downward" and use "reduce upward" in a similar sense. We also use "continue upward". Only in doubt, the clumsy expression "harmonically continue upward" should be employed. Also "analytical continuation" is used. It all means the same. In the present context, confusion is hardly possible.

Hence we see why gravity anomalies Δg *at ground level* may be used for f in (8–58), whereas the equivalent expression (2–394) was originally derived for gravity anomalies *at sea level.* Since $\Delta g^{\text{harmonic}}$ and Δg differ only by terms of the order of h, the difference between using $\Delta g^{\text{harmonic}}$ or Δg in (8–62) causes only an error of the order of h^2, which is negligible in the linear approximation.

Analytical continuation: historical remarks

The use of analytical continuation has an interesting history. It was first considered as a possibility by Molodensky himself, already before 1945, but he soon rejected this method! Molodensky was a profound mathematician, with a high regard for mathematical rigor. He would not be satisfied with intuitive heuristic approaches so common in mathematical physics, also in the present book.

In fact, the analytical continuation of the external gravitational potential into the interior of the earth's masses is very likely to become singular at some points. As a serious mathematician, Molodensky rejected the use of singular functions for regular purposes.

Still, analytical continuation continued to exert an irresistable fascination because its use is so easy. It was rediscovered around 1960 by A. Bjerhammar. At the General Assembly of the International Union of Geodesy and Geophysics in Berkeley, California, in 1963, one of the authors (H.M.) talked to Bjerhammer about these difficulties, but Bjerhammar refused to take them seriously. After a long discussion he convinced H.M. that analytical continuation was rigorously possible for discrete boundary data (all our terrestrial gravity measurements are discrete) and approximately possible for continuous boundary data.

This admittedly intuitive thinking was made rigorous by the idea of Krarup (1969) that Runge's theorem, well known for approximation of analytical functions of a complex variable, should be applied to the problem of analytical continuation of harmonic functions in space. Runge's theorem, in the form of Krarup, loosely speaking says that, even if the external geopotential cannot be regularly continued from the earth surface S into its interior, it can be made continuable by an arbitrarily small change of the geopotential at S. Another historical remark: the Krarup–Runge theorem for harmonic functions in space goes back at least to Szegö and to Walsh (both around 1929), cf. Frank and Mises (1930: pp. 760–762). It is always dangerous to talk about priorities! A detailed discussion will be found in Moritz (1980 a: Sects. 6 to 8).

More on the validity of this method

Let us summarize. The presupposition of this method is that the earth's external gravitational potential can be continued, as a regular harmonic function, analytically down to sea level. This is the case if and only if it is possible to shift the masses outside the ellipsoid into its interior in such a way that the potential outside the earth remains unchanged or, in other words, if the analytical continuation of the disturbing potential T is a regular function everywhere between the earth's surface and the ellipsoid. Thus, the question arises whether the external potential can be analytically continued down to sea level.

Rigorously, as we have just remarked, the answer must be in the negative, in view of the irregularities of topography (Molodenski et al. 1962: p. 120; Moritz 1965: Sect. 6.4). This fact is also related to the divergence at the earth's surface of the spherical-harmonic expansion for the external potential (Sect. 2.5).

However, by Krarup–Runge's theorem, the analytical continuation of the external potential down to sea level is possible *with sufficient accuracy for all practical purposes.* Actually it is possible with any accuracy you wish; if you are not satisfied with 1 mgal, prescribe 10^{-3} mgal or 10^{-1000} mgal!

Bjerhammar has pointed out that the assumption of a complete continuous gravity coverage at every point of the earth's surface, from which the above negative answer follows, is unrealistic because we can measure gravity only at discrete points. If the purpose of physical geodesy is understood as the *determination of a gravity field that is compatible with the given discrete observations*, then it is always possible to find a potential that can be analytically continued down to the ellipsoid. This is the theoretical basis for *least-squares collocation*.

Here we need only one result: **Do not worry about analytical continuation! It is always possible with an arbitrarily small error being not equal to** 0 (though not for one being 0).

So, in the same year 1969, Marych and Moritz independently found an elementary solution by analytical continuation in the form of an infinite series denoted as "Molodensky series". Details can be found in Moritz (1980 a): The original form of Molodensky's series obtained by solving an integral equation is found in Sect. 45. Pellinen's equivalence proof that *the simple "analytical continuation solution" and Molodensky's integral equation solution are equivalent* (that means, the series are termwise equal!) is found in Sect. 46.

We remark that analytical continuation is a purely mathematical concept independent of the density of the topographic masses. Thus, it is not an "introduction of gravity reduction through the backdoor", which would be contrary to the spirit of Molodensky's theory.

8.6.5 Another perspective

Consider Fig. 8.6. Let us assume that the analytical downward continuation of Δg to the sea level surface has been performed, obtaining $\Delta g^{\text{harmonic}}$. The sea-level anomalies $\Delta g^{\text{harmonic}}$ then generate, on the physical surface of the earth, a field of gravity anomalies Δg that is identical with the actual gravity anomalies on the earth's surface as obtained from observation. Therefore, the gravity anomalies that they generate outside the earth must also be identical

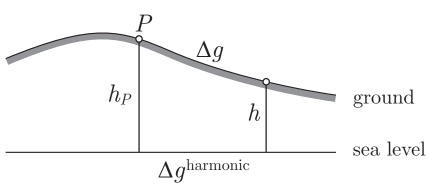

Fig. 8.6. Free-air anomalies at ground level, Δg, and at sea level, $\Delta g^{\text{harmonic}}$

with the actual gravity anomalies outside the earth, since the function $r\,\Delta g$ is harmonic according to Sect. 2.14.

(Remark: we are consistently using the notation Δg for ground level, $\Delta g^{\text{harmonic}}$ for sea level, and Δg^* for point level; see Fig. 8.5.)

It follows that the harmonic function T that is produced by $\Delta g^{\text{harmonic}}$ according to Pizzetti's generalization (2–302) of Stokes' formula

$$T(r,\vartheta,\lambda) = \frac{R}{4\pi} \iint_\sigma \Delta g^{\text{harmonic}}\, S(r,\psi)\, d\sigma \tag{8–74}$$

is identical with the actual disturbing potential of the earth *outside and on its surface.*

Applications

Assume that we got in some way (e.g., by the Taylor series mentioned above or by collocation to be treated in Chap. 10 or by a high-resolution gravitational field from satellite observations) the downward continuation $\Delta g^{\text{harmonic}}$ to sea level. Then we can compute the external gravity field, its spherical harmonics, etc., rigorously by means of the conventional formulas of Chaps. 2 and 6, provided we use $\Delta g^{\text{harmonic}}$ rather than Δg in the relevant formulas. For instance, the coefficients of the spherical harmonics of the gravitational potential may be obtained by expanding the function $\Delta g^{\text{harmonic}}$ according to Sect. 1.9 together with Sect. 1.6. If we wish to compute the height anomaly ζ at a point P at ground level, we must remember that P lies *above* the ellipsoid, so that the formulas for the external gravity field are to be applied. By Bruns' formula $\zeta = T/\gamma_0$ (8–50), we get

$$\zeta = \frac{R}{4\pi\,\gamma_0} \iint_\sigma \Delta g^{\text{harmonic}}\, S(r,\psi)\, d\sigma\,, \tag{8–75}$$

where $r = R + h$ and h is the topographic height of P in some sense of Chap. 4. (We do not need it very accurately, but it means that h is *formally* added to the constant radius R of the mean terrestrial sphere, which has no real-world geometric interpretation!) Cf. Eq. (6–57). The function $S(r,\psi)$ is expressed by (2–303), (6–22) or (6–35). Similarly, ξ and η, being deflections of the vertical above sea level, must be computed by (6–41) and the second and third equation of (6–30). This gives

$$\begin{aligned} \xi &= \frac{t}{4\pi\,\gamma_0} \iint_\sigma \Delta g^{\text{harmonic}}\, \frac{\partial S(r,\psi)}{\partial\psi}\, \cos\alpha\; d\sigma\,, \\ \eta &= \frac{t}{4\pi\,\gamma_0} \iint_\sigma \Delta g^{\text{harmonic}}\, \frac{\partial S(r,\psi)}{\partial\psi}\, \sin\alpha\; d\sigma\,, \end{aligned} \tag{8–76}$$

where $\partial S(r, \psi)/\partial \psi$ is expressed by the second equation of (6–32) or the second equation of (6–36). The linear approximation of (8–74) is evidently equivalent to (8–53).

This indirect procedure, downward continuation to sea level and again upward continuation to ground level or above, has the advantage that only the conventional spherical formulas are needed; yet at the same time the irregularities of the earth's topography are fully taken into account. The downward continuation of Δg need be performed only once; the resulting anomalies $\Delta g^{\text{harmonic}}$ may be stored and used for all further computations.

Just as Δg is related to $\Delta g^{\text{harmonic}}$ by analytical continuation, so are ζ and N^{harmonic}, the height of a "harmonic geoid". A final and hopefully instructive and not too difficult review will be found in Sect. 8.15.

An elementary explanation from daily life

Generally, "analytical continuation" means continuation by the same mathematical formula: Taylor series, Laplace equation, or even an elementary explicit equation.

Let us illustrate the meaning of analytical continuation by means of an almost trivial example from everyday life (Fig. 8.7). A person is driving a car along a road which at first is completely straight; at point B, however, it suddenly turns into a circular curve. Thus, our person first drives along the straight segment of the road. Unfortunately, he is tired and sleepy just when the straight road suddenly turns into a circular curve. Thus, our sleepy driver fails to turn the steering wheel and goes straight ahead, the car leaving the road. Fortunately, the slope is mild, the driver immediately takes control again and manages to bring the car to a stop at C' without major damages. The driver (one of the authors of this book) has even found the experience an excellent example to illustrate analytical continuation in his courses!

The gravitational potential corresponds to the road ABC, which, after some idealization, can be considered *"piecewise analytic"*, consisting of the straight line AB and the circular arc BC. The transition from the straight line to the circle is continuous and differentiable at B, but the curvature

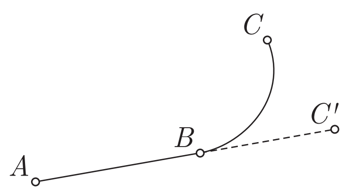

Fig. 8.7. An illustration of analytical continuation

changes discontinuously from 0 to $1/R$, where R is the radius of the circular arc. Therefore, the function "road" is continuous and continuously differentiable but has a discontinuous second derivative at point B, just as the function "gravitational potential" is everywhere continuous and continuously differentiable but has discontinuous second derivatives at the earth surface as we have seen in Sect. 1.2. The straight line has the equation $y'' = 0$ (which is the "one-dimensional Laplace equation"!), corresponding to the external potential satisfying $\Delta V = 0$. Thus, neither the "function road" nor the "function gravitational potential" may be considered an everywhere analytical function, but each may be said to consist of a "linear" or "harmonic" piece ($y'' = 0$ or $\Delta V = 0$: Laplace, respectively) and a "nonlinear" piece ($y'' \neq 0$ or $\Delta V = -4\pi G \varrho$: Poisson). For the road, the analytical continuation is the straight line for which $y'' = 0$ even beyond point B, the path followed by the car without action of the sleepy driver towards C', and for the potential it is a function satisfying $\Delta V_{\text{analytical continuation}} = 0$ even in the interior of the earth.

8.7 Deflections of the vertical

The effect of Molodensky-type corrections is even much more important on the deflections of the vertical ξ, η than on the height anomalies ζ. This is shown by their order of magnitude in high mountains: the Molodensky correction for the height anomaly might be of the order of 0.3 m, whereas for vertical deflections they may be on the order of 0.3 arc seconds, which corresponds to 10 m (1 arc second corresponds to 30 m in position). *The difference is more than one order of magnitude!*

The consideration of a Molodensky type of correction to the deflections of the vertical is easiest by using analytical continuation to point level (Sect. 8.6). Differentiating (8–57) in north-south and east-west direction, we get the corresponding Vening Meinesz equations

$$\begin{aligned} \xi &= \frac{1}{4\pi\,\gamma_0} \iint_\sigma \left[\Delta g - \frac{\partial \Delta g}{\partial h}\,(h - h_P)\right] \frac{dS}{d\psi} \cos\alpha \, d\sigma \,, \\ \eta &= \frac{1}{4\pi\,\gamma_0} \iint_\sigma \left[\Delta g - \frac{\partial \Delta g}{\partial h}\,(h - h_P)\right] \frac{dS}{d\psi} \sin\alpha \, d\sigma \,. \end{aligned} \tag{8–77}$$

Its geometrical interpretation is analogous to that of (8–57). The gravity anomalies Δg are "reduced" to the level of point P so that we obtain

$$\Delta g^{\text{harmonic}} = \Delta g - \frac{\partial \Delta g}{\partial h}\,(h - h_P)\,. \tag{8–78}$$

Since these anomalies *refer to a level surface*, Vening Meinesz' formula can now be applied directly and gives (8–77).

Relation with the ellipsoidal geodetic coordinates

The deflection components ξ and η as given by the above expressions represent the deviation of the actual plumb line from the normal plumb line at the ground point P. Therefore, they are defined by

$$\begin{aligned} \xi &= \Phi - \varphi^* , \\ \eta &= (\Lambda - \lambda^*)\, \cos\varphi . \end{aligned} \tag{8–79}$$

The symbols Φ and Λ represent the astronomical coordinates of P referred to the ground. The symbols φ^* and λ^* represent the *"normal coordinates"* of P, defining the direction of the normal plumb line at P; they are not identical with the ellipsoidal coordinates φ and λ of P, which are the coordinates of the foot point Q_0 of the straight perpendicular to the ellipsoid (Fig. 8.8).

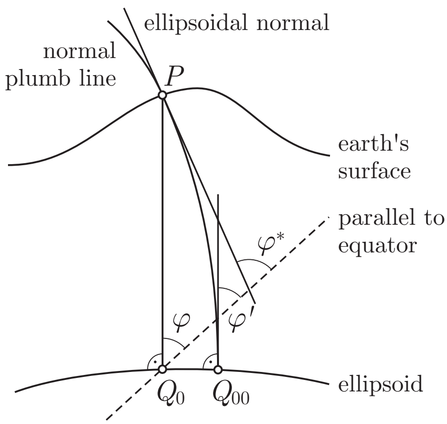

Fig. 8.8. Normal latitude φ^* and ellipsoidal latitude φ

The normal coordinates of P, φ^* and λ^*, differ from the normal coordinates of Q_{00}, φ' and λ', by the correction for the normal curvature of the plumb line (see Sect. 5.15). Formula (5–147) gives

$$\begin{aligned} \varphi^* &= \varphi' + f^* \frac{h}{R} \sin 2\varphi , \\ \lambda^* &= \lambda' . \end{aligned} \tag{8–80}$$

Because of the rotational symmetry, we have rigorously

$$\lambda' = \lambda , \tag{8–81}$$

since Q_0 and Q_{00} lie on the same ellipsoidal meridian. Furthermore, even in extreme cases the distance between Q_0 and Q_{00} can never exceed a few centimeters. For this reason, we may also set

$$\varphi' = \varphi \tag{8–82}$$

without introducing a perceptible error. Hence, we can identify φ' and λ' with φ and λ, which are the *ellipsoidal coordinates* of P according to Helmert's projection (Sect. 5.5). Therefore, we may replace the above equations for φ^* and λ^* by

$$\begin{aligned} \varphi^* &= \varphi + f^* \frac{h}{R} \sin 2\varphi \,, \\ \lambda^* &= \lambda \,. \end{aligned} \tag{8–83}$$

Introducing the deflection components according to Helmert's projection, defined as

$$\begin{aligned} \xi_{\text{Helmert}} &= \Phi - \varphi \,, \\ \eta_{\text{Helmert}} &= (\Lambda - \lambda) \cos\varphi \,, \end{aligned} \tag{8–84}$$

we see that they are related to ξ and η by the equations

$$\begin{aligned} \xi_{\text{Helmert}} &= \xi + f^* \frac{h}{R} \sin 2\varphi \,, \\ \eta_{\text{Helmert}} &= \eta \,. \end{aligned} \tag{8–85}$$

Therefore, ξ and ξ_{Helmert} differ by the normal reduction for the curvature of the plumb line,

$$-\delta\varphi_{\text{normal}} = f^* \frac{h}{R} \sin 2\varphi \,. \tag{8–86}$$

The deflection components ξ_{Helmert} and η_{Helmert} are used in astrogeodetic computations; ξ and η are those obtained gravimetrically from formulas such as (8–77) and (8–88) below.

These relations are *mathematically* quite analogous to the corresponding equations (5–138) for the conventional method using the geoid, but now, with the use of the *normal* curvature, the once formidable obstacle of the correction for plumb-line curvature *practically* belongs to the past.

Remark on accuracy

With Molodensky's theory, the accuracy problem mentioned at the end of Sect. 2.21 even aggravates, because in a mountainous terrain it is almost impossible to compute the Molodensky corrections with an accuracy of $0.03''$ (say), so that these observations cannot be directly used for precise horizontal positions.

Astronomical field observations for latitude, longitude, and azimuth have an accuracy around $0.3''$, which is sufficient for classical trigonometric net computation and astrogeodetic observation of the geoid (Sect. 5.14).

8.8 Gravity disturbances: the GPS case

The basic fact is that for gravity disturbances the derivation of "Molodensky corrections" g_n is identical to the Δg case. The reason is that the gravity disturbance δg has exactly the same analytical behavior as the gravity anomaly Δg since $r\,\delta g$, as a function in space, is harmonic together with $r\,\Delta g$. Thus, the arguments are literally the same, only Δg has to be replaced by δg, and Stokes' formula must be replaced by the Neumann–Koch formula (8–47) and similarly for Vening Meinesz' formula.

Therefore, we obtain

$$\zeta = \frac{R}{4\pi\,\gamma_0} \iint_\sigma \delta g\, K(\psi)\, d\sigma + \sum_{n=1}^{\infty} \frac{R}{4\pi\,\gamma_0} \iint_\sigma g_n\, K(\psi)\, d\sigma\,, \tag{8–87}$$

$$\begin{aligned} \xi &= \frac{1}{4\pi\,\gamma_0} \iint_\sigma \delta g\, \frac{dK}{d\psi} \cos\alpha\, d\sigma + \sum_{n=1}^{\infty} \frac{1}{4\pi\,\gamma_0} \iint_\sigma g_n\, \frac{dK}{d\psi} \cos\alpha\, d\sigma\,, \\ \eta &= \frac{1}{4\pi\,\gamma_0} \iint_\sigma \delta g\, \frac{dK}{d\psi} \sin\alpha\, d\sigma + \sum_{n=1}^{\infty} \frac{1}{4\pi\,\gamma_0} \iint_\sigma g_n\, \frac{dK}{d\psi} \sin\alpha\, d\sigma\,. \end{aligned} \tag{8–88}$$

For the "Vening Meinesz GPS formula" (8–88), we find by differentiation of (8–49):

$$\frac{dK}{d\psi} = -\frac{1}{2}\,\frac{\cos(\psi/2)}{\sin^2(\psi/2)}\,\frac{1}{1+\sin(\psi/2)}\,. \tag{8–89}$$

The correction terms g_n are evaluated recursively by

$$g_n = -\sum_{r=1}^{n} z^r\, L_r(g_{n-r})\,, \tag{8–90}$$

but now we start from

$$g_0 = \delta g\,. \tag{8–91}$$

We only have to replace Δg by δg and $S(\psi)$ by $K(\psi)$. The operators L remain the same.

Let us summarize again our trick for solving the modern boundary-value problems (Molodensky and Koch). It is difficult to directly work with the complicated earth's surface S. Therefore, by analytical continuation of Δg or

δg, respectively, we reduce these complicated problems to the corresponding spherical problems, for which the solution is simple and well known.

The similarity of the Molodensky series for the Molodensky problem, on the one hand, and for the GPS boundary problem, on the other hand, is very clear because Δg and δg have the same analytical and geometric structure.

At the same time, this similarity is very surprising since the two underlying boundary problems are mathematically quite different, as we have seen in Sect. 8.3 (compare Eqs. (8–12) and (8–13)). Nonetheless, (8–87) does give the potential as (8–13) requires: by Bruns' theorem, which is the omnipresent link between geometry and physics, we have

$$T = \gamma \zeta \, . \tag{8–92}$$

Then

$$W = U + T \tag{8–93}$$

is the geopotential required by (8–13), and

$$C = W_0 - W \tag{8–94}$$

is the geopotential number, the physical measure of height above sea level, conventionally obtained by the cumbersome method of leveling, but now computed in a direct way from gravity data. This is the physical, more general, equivalent of the geometric determination of the normal height by $H^* = h - \zeta$, according to Eq. (8–31).

It can be shown that, in the linear approximation, the Molodensky correction for the gravity disturbance has the same form as for the gravity anomaly and can for each quantity be computed using either Δg or δg.

The formulas for the Molodensky corrections and their numerical values are the same to the linear approximation.

All this shows the power of Molodensky's approach even in problems he never treated himself.

8.9 Gravity reduction in the modern theory

In Sect. 8.2, we have considered gravity reductions from the point of view of the determination of the geoid. It is quite remarkable that these reductions, such as the Bouguer or the isostatic reduction, can also be incorporated into the new method of direct determination of the earth's physical surface, although with essentially changed meaning (Pellinen 1962; Moritz 1965: Sect. 5.2).

Let the masses outside the geoid be removed or moved inside the geoid, as described in Sect. 8.2, and consider the effect of this procedure on quantities *referred to the ground.*

We denote the changes in potential and in gravity by δW and δg; then the new values at ground will be

$$\begin{aligned} W^c &= W - \delta W\,, \\ g^c &= g - \delta g\,. \end{aligned} \tag{8–95}$$

(It is clear that δg here is not the gravity disturbance!) The disturbing potential $T = W - U$ becomes

$$T^c = T - \delta W\,. \tag{8–96}$$

The physical surface S as such will remain unchanged, but the telluroid Σ will change, because its points Q are defined by $U_Q = W_P$, and the potential W at any surface point P will be affected by the mass displacements according to (8–95). The distance $Q\,Q^c$ between the original telluroid Σ and the changed telluroid Σ^c (Fig. 8.9) is given by

$$Q\,Q^c = \frac{\delta W}{\gamma} \tag{8–97}$$

according to Bruns' theorem. This is identical with the variation of the height anomaly ζ, so that

$$\delta\zeta = \zeta - \zeta^c = \frac{\delta W}{\gamma}\,. \tag{8–98}$$

Normal gravity γ on the telluroid Σ becomes on the changed telluroid Σ^c

$$\gamma^c = \gamma + \frac{\partial\gamma}{\partial h}\,\delta\zeta = \gamma + \frac{1}{\gamma}\,\frac{\partial\gamma}{\partial h}\,\delta W\,, \tag{8–99}$$

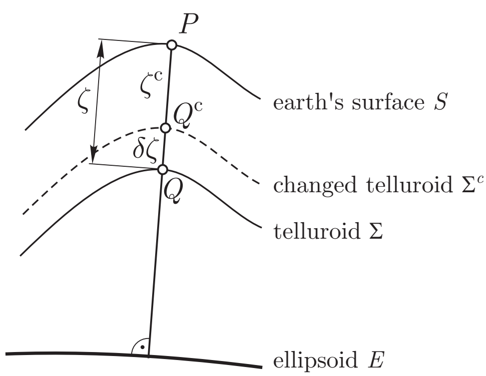

Fig. 8.9. Telluroid before and after gravity reduction, Σ and Σ^c

so that the new gravity anomaly will be

$$\Delta g^{\mathrm{c}} = g^{\mathrm{c}} - \gamma^{\mathrm{c}} = (g - \delta g) - \left(\gamma + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}\,\delta W\right) \tag{8–100}$$

or

$$\Delta g^{\mathrm{c}} = \Delta g - \delta g - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}\,\delta W \,. \tag{8–101}$$

The reduced gravity anomaly Δg^{c} consists of the free-air anomaly (in the Molodensky sense) Δg and two reductions:

1. the direct effect, $-\delta g$, of the shift of the outer masses on g; and
2. the "indirect effect",

$$-\frac{1}{\gamma}\frac{\partial \gamma}{\partial h}\,\delta W \,, \tag{8–102}$$

   of this shift on γ, because of the change of the telluroid to which γ refers.

Let us repeat once more that all these anomalies Δg^{c} refer to the physical surface of the earth, to "ground level"!

If the masses outside the geoid are completely removed, then Δg^{c} is a Bouguer anomaly; if the outer masses are shifted vertically downward according to some isostatic hypothesis, then Δg^{c} is an isostatic anomaly, etc. In this way we may get a "ground equivalent" for each conventional gravity reduction. The two are always related by analytical continuation. See below for the isostatic anomalies; for analytical continuation see Sect. 8.6.

Now we may describe the determination of the height anomalies ζ in a way that is similar to the corresponding procedure for the geoidal undulations N of Sect. 8.2:

1. The masses outside the geoid are, by computation, removed entirely or else moved inside the geoid; W and g change to W^{c} and g^{c} according to (8–95).
2. The point at which normal gravity is computed is moved from the ellipsoid upward to the telluroid point Q.
3. The indirect effect, the distance $Q\,Q^{\mathrm{c}} = \delta\zeta$, is computed by (8–98).
4. The point to which normal gravity refers is now moved from the point Q of the telluroid Σ to the point Q^{c} of the changed telluroid Σ^{c}, according to (8–99).
5. The changed height anomalies ζ^{c} are computed from the "reduced" gravity anomalies Δg^{c} (8–101) by any solution of Molodensky's problem, such as Eq. (8–57) or (8–68).

6. Finally, the original height anomalies ζ are obtained by considering the indirect effect according to

$$\zeta = \zeta^{\mathrm{c}} + \delta\zeta \,. \tag{8–103}$$

The purpose of this somewhat complicated procedure is to make use of the well-known advantages of Bouguer and isostatic anomalies. The Bouguer anomalies, and even more so the isostatic anomalies, are *smoother and more representative* than the free-air anomalies and can, therefore, be interpolated more easily and more accurately.

The isostatic gravity anomalies Δg^{c} in the new sense are thus quite analogous to the conventional isostatic anomalies; accordingly for any other type of gravity reduction. The difference is that now the isostatic anomalies, etc., *refer to the physical surface of the earth* as well as the free-air anomalies. If the isostatic anomalies in this new sense are analytically continued from the earth's surface down to the geoid, then isostatic anomalies in the conventional sense are obtained. Nowadays, in view of the "remove-restore principle", one speaks usually of *topographic-isostatic reduction* while continuing to speak of isostatic anomalies.

Hence, the isostatic anomalies according to the conventional definition (at sea level) and those according to the new definition (at ground level) are related through analytical continuation. This fact leads to two conclusions. First, the difference between the isostatic anomalies according to these two definitions will be small, because the distance along which this analytical continuation is made is only the height above sea level and because the isostatic reduction achieves a strong smoothing of the anomalous gravity field. This difference is considerably smaller than the corresponding difference between free-air anomalies at ground level and at sea level. This fact clearly provides a computational advantage if isostatic anomalies are used in a formula such as (8–74).

Second, we obtain a relation between the conventional and the modern use of gravity reduction if the method of downward continuation, as discussed in the preceding section, is applied for obtaining the height anomalies. As we have just seen, the gravity anomalies $\Delta g^{\mathrm{c}*}$ at sea level, obtained by downward continuation of the isostatic ground-level anomalies Δg^{c}, are identical with the isostatic anomalies in the conventional sense. Hence, we obtain on the one hand the height anomalies by

$$\zeta = \frac{R}{4\pi\,\gamma} \iint_{\sigma} \Delta g^{\mathrm{c}*}\, S(R+h,\psi)\, d\sigma + \left(\frac{\delta W}{\gamma}\right)_{\mathrm{ground}} \tag{8–104}$$

according to (8–75) and (8–103), and on the other hand the geoidal undulations by

$$N = \frac{R}{4\pi\,\gamma_0} \iint\limits_{\sigma} \Delta g^{c*}\, S(\psi)\, d\sigma + \left(\frac{\delta W}{\gamma}\right)_{\text{geoid}} \tag{8–105}$$

according to the ordinary Stokes formula applied to Δg^{c*} and (8–5). Since the height anomalies refer to the elevation h, the function $S(R{+}h, \psi)$ replaces in (8–104) the original function of Stokes $S(\psi) \equiv S(R, \psi)$, which occurs in (8–105) because the geoidal undulation refers to zero elevation. We could use γ_0 in (8–104) as well. Summarizing, we have the following steps:

1. Computation of the free-air anomaly at ground level, Δg, according to (8–23).
2. Computation of the isostatic anomaly at ground level, Δg^c, according to (8–101).
3. Downward continuation of Δg^c by (8–54), where Δg and $\Delta g^{\text{harmonic}}$ are replaced by Δg^c and Δg^{c*}. The resulting isostatic anomalies at sea level, Δg^{c*}, may now be used for two purposes: either for

4a. the determination of the physical surface of the earth according to (8–104), or for

4b. the determination of the geoid according to (8–105).

An error in the assumed density of the masses below the earth's surface affects the geoidal undulations as determined from (8–105) but does not influence the height anomalies resulting from (8–104). This is clear because a wrong guess of the density means only that the masses above sea level are not completely removed, which is no worse than not removing them at all when using free-air anomalies.

This method is of particular interest for practical computations, as we will see later. It has become popular by the name *"remove-restore method"*, invented by K. Colic and others, see Sect. 11.1.

An almost final remark on free-air reduction

The apparently so simple topic of free-air reduction in reality is formidably complex and complicated. Therefore, it is not possible to treat it in one block. The problem is rather like a mountain which can only be investigated by accessing it from various sides. An initial glance has been given as early as in Chap. 3, and the reader is asked to return to the paragraph "The many facets of free-air reduction" in Sect. 3.9. Now it is much easier to understand the remarks made there. What we now understand as harmonic continuation offers a possibility to *interpret free-air reduction as a mass-transporting gravity reduction*: the topographic masses are transported into the interior of the earth in such a way that the exterior potential remains unchanged. This is not unlike the Rudzki reduction, where *the geoid* remains unchanged. Whereas the Rudzki reduction is, however, "constructive" in the sense that a way of performing it can be described, our present interpretation of free-air reduction as harmonic continuation is nonconstructive, it is an "improperly posed" inverse problem; cf. Anger and Moritz (2003) and www.inas.tugraz.at/forschung/InverseProblems/AngerMoritz.html, as well as Fig. 8.10.

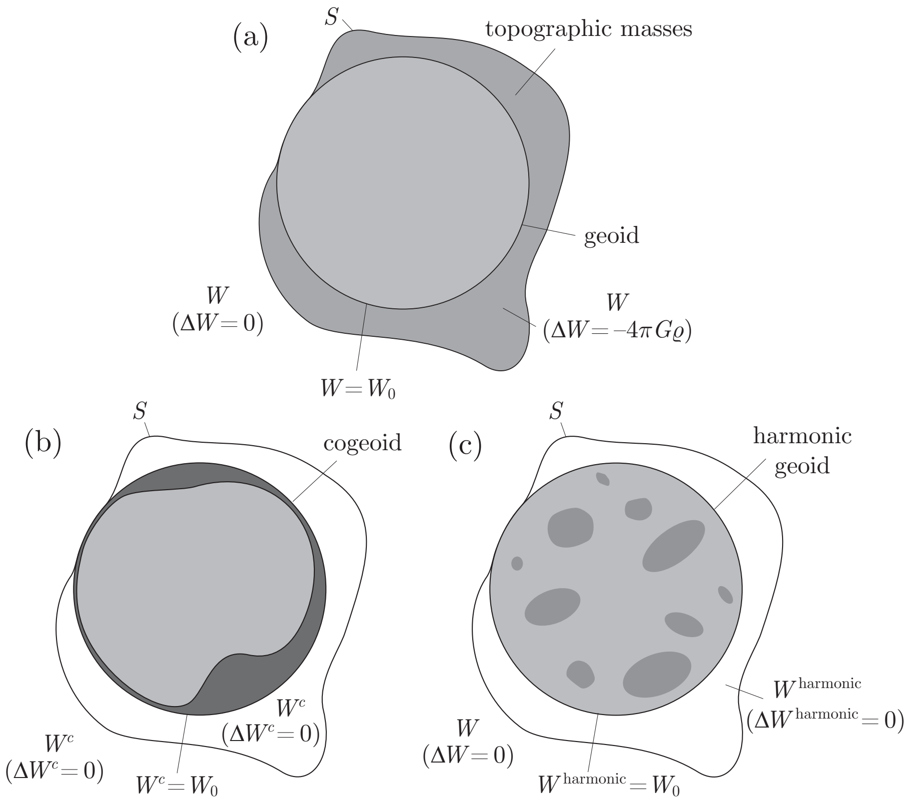

Fig. 8.10. (a) Geoid and topographic masses, (b) mass displacement in gravity reduction, (c) "ill-defined" mass displacement in free-air reduction as harmonic continuation

Important remark

The isostatic gravity anomalies and the topographic-isostatically reduced deflections of the vertical (Sect. 8.14) are fundamental for least-squares collocation in mountain areas (Sect. 11.2). **Thus, the spatial approach due to Molodensky is basic even for least-squares collocation!**

Exercise

Collecting all these remarks into a separately readable paper on the various aspects of free-air reduction would be a nice task for a seminar work. The present authors offer a prize of Euro 500, the "Molodensky Prize", to the first excellent review paper on this topic.

8.10 Determination of the geoid from ground-level anomalies

We have seen that it is possible to determine the physical surface of the earth by means of the height anomalies ζ, and the direction of the plumb line on it by means of the deflection components ξ and η, from free-air anomalies referred to the ground. If we plot the orthometric height H downward along the plumb line, starting from the physical surface, then the locus of the points so obtained will be the geoid (Fig. 8.11).

This geometrical idea may be formulated analytically in the following way. Conventionally, the height h above the ellipsoid is given by

$$h = H + N ; \tag{8–106}$$

according to the modern theory, by

$$h = H^* + \zeta . \tag{8–107}$$

From these two equations we get

$$N - \zeta = H^* - H . \tag{8–108}$$

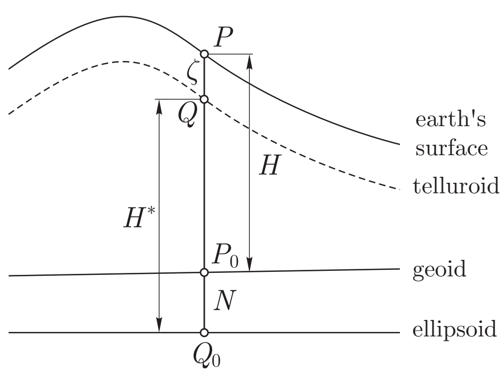

Fig. 8.11. Geoid at a depth H below the earth's surface

This means that the difference between the geoidal undulation N and the height anomaly ζ is equal to the difference between the normal height H^* and the orthometric height H. Since ζ is also the undulation of the quasigeoid, this difference is also the distance between geoid and quasigeoid.

According to Sect. 4.5, the two heights are defined by

$$H = \frac{C}{\bar{g}}, \qquad H^* = \frac{C}{\bar{\gamma}}, \tag{8–109}$$

where C is the geopotential number, $\bar{g}$ is the mean gravity along the plumb line between geoid and ground, and $\bar{\gamma}$ is the mean normal gravity along the normal plumb line between ellipsoid and telluroid. By eliminating C between these two equations, we readily find

$$H^* - H = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H, \tag{8–110}$$

which is also the distance between the geoid and the quasigeoid, see (8–108); hence

$$N = \zeta + \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H. \tag{8–111}$$

The height anomaly ζ may be expressed, for instance, by Molodensky's formula (8–57). Then we obtain

$$N = \frac{R}{4\pi\gamma_0} \iint_\sigma \Delta g\, S(\psi)\, d\sigma + \frac{R}{4\pi\gamma_0} \iint_\sigma g_1\, S(\psi)\, d\sigma + \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H, \tag{8–112}$$

where g_1 is the term (8–62). Thus N is given by Stokes' integral, *applied to free-air anomalies at ground level*, and two small corrections, where

1. the term containing g_1 represents the effect of topography;

2. the term containing $\bar{g} - \bar{\gamma}$ represents the distance between the geoid and the quasigeoid.

If we neglect these two corrections, then the geoidal undulations N are given by Stokes' integral using free-air anomalies. This was first noted by Stokes in 1849. A new approach by Jeffreys (1931) by means of Green's identities started several developments which culminated in the work of Molodensky and others.

The advantage of this method for the determination of N is that the density of the masses above sea level enters only indirectly, as an effect on the orthometric height H through the mean gravity $\bar{g}$, which must be computed by a Prey reduction (Sect. 3.5). Hence, as far as errors in the density are concerned, the geoidal undulation N as obtained by this method is as accurate as the orthometric height.

As a matter of fact, the gravity anomaly Δg in this method refers to ground level; it is the difference between gravity at ground and normal gravity at the telluroid. Instead of using directly this free-air anomaly, we may also use other gravity anomalies – for instance, the isostatic anomaly in the sense of Sect. 8.9.

To repeat a simple but fundamental principle: Δg, δg, ξ, η, ζ as obtained by Molodensky's theory **primarily** *always refer to the physical earth's surface and not to sea level!*

8.11 A first balance

The new methods described in this chapter are primarily intended for the determination of the physical surface of the earth, but they are also well suited for the determination of the geoid (Sect. 8.10). Their essential feature is that the gravity anomalies *now* refer to the ground, whether we deal with free-air anomalies or with isostatic or other similarly reduced gravity anomalies (Sect. 8.9).

The immediate result is the height anomaly ζ, the separation between the geopotential and the corresponding spheropotential surface at ground level. By plotting the height anomalies above the ellipsoid, we get the quasigeoid. This geoid-like surface has no physical significance, but it furnishes a convenient visualization of the height anomalies. By plotting the orthometric height from the earth's surface vertically downward, we obtain the geoid.

It is instructive to compare the geoid and the quasigeoid. The geoidal undulation N and ζ, the undulation of the quasigeoid, are related by (8–111), or

$$N - \zeta = \frac{\bar{g} - \bar{\gamma}}{\bar{\gamma}} H = H^* - H \,. \tag{8–113}$$

The term $\bar{g} - \bar{\gamma}$ is approximately equal to the Bouguer anomaly; this may be seen by using (4–32) for γ together with

$$\bar{\gamma} \doteq \gamma - \frac{1}{2} \frac{\partial \gamma}{\partial h} H \,. \tag{8–114}$$

The quantity $\bar{\gamma}$ in the denominator can be replaced by our usual constant γ_0. Since the Bouguer anomaly is rather insensitive to local topographic irregularities, the coefficient is locally constant so that there is approximately a linear relation between ζ and the local irregularities of the height H. In other words, the *quasigeoid mirrors the topography* (Fig. 8.12).

Fig. 8.12. Quasigeoid

To get a quantitative estimate of the difference $N - \zeta$, we again use the fact that

$$\frac{\bar{g}-\bar{\gamma}}{\bar{\gamma}} \doteq \frac{\Delta g_B}{981 \text{ gal}} \doteq 10^{-3} \Delta g_B \,, \tag{8–115}$$

where Δg_B is the Bouguer anomaly in gal, so that

$$(\zeta - N)_{[\mathrm{m}]} \doteq -\Delta g_{B\,[\mathrm{gal}]} \cdot H_{[\mathrm{km}]} \,. \tag{8–116}$$

Since Δg_B is usually negative on the continents, the differences $\zeta - N$ are usually positive there. In other words, the height anomaly ζ is in general greater than the corresponding geoidal undulation N on land. We have $\zeta = N$ on the oceans. If $\Delta g_B = -100\,\mathrm{mgal} = -0.1\,\mathrm{gal}$ and $H = 1\,\mathrm{km}$, then

$$\zeta - N = 0.1 \text{ m} \,. \tag{8–117}$$

Furthermore, the Bouguer anomaly depends on the *mean* elevation of the terrain, decreasing approximately by 0.1 gal per 1 km average elevation. Assuming as a rough estimate, which may be verified by inspecting maps of Bouguer anomalies,

$$\Delta g_{B\,[\mathrm{gal}]} = -0.1\, H^{\mathrm{av}}_{[\mathrm{km}]} \,, \tag{8–118}$$

we obtain

$$(\zeta - N)_{[\mathrm{m}]} \doteq +0.1\, H^{\mathrm{av}}_{[\mathrm{km}]}\, H_{[\mathrm{km}]} \,, \tag{8–119}$$

where H is the height of the station and H^{av} is an average height of the area considered. We see that the difference $\zeta - N$ increases faster than the elevation, almost as the square of the elevation. As a matter of fact, this formula is suited only to give an idea of the order of magnitude (see also Sect. 11.3).

Since $\zeta - N = H - H^*$, the approximate formulas given above may also be used to estimate the differences between the orthometric height H and the normal height H^*.

A theoretically important point is that the quasigeoid can be determined without hypothetical assumptions concerning the density, but not so the geoid. The avoidance of such assumptions has been the guiding idea of Molodensky's research. However, orthometric heights are but little affected by errors in density. The error in H due to the imperfect knowledge of the density hardly ever exceeds 1–2 decimeters even in extreme cases (Sect. 4.3). It is presumably smaller than the inaccuracy of the corresponding ζ even with very good gravity coverage, because of inevitable errors of interpolation, etc. If, therefore, the method of Sect. 8.10 is used, the geoid can be determined with virtually the same accuracy as the quasigeoid. Note that it is theoretically even possible to eliminate completely the errors arising from the use of the geoid (Moritz 1962, 1964). Thus, we may well retain the geoid with its physical significance and its other advantages.

How much do Molodensky's formulas differ from the corresponding equations of Stokes and Vening Meinesz? The deviation of ζ from the result of the original Stokes formula is given by the equivalent expressions

$$\zeta_1 = \frac{R}{4\pi\,\gamma_0}\iint_\sigma g_1\,S(\psi)\,d\sigma \quad \text{or} \quad \zeta_1 = -\frac{R}{4\pi\,\gamma_0}\iint_\sigma \frac{\partial \Delta g}{\partial h}\,(h-h_P)\,S(\psi)\,d\sigma \tag{8–120}$$

according to Eqs. (8–62) and (8–66). This correction may even be smaller than the difference $\zeta - N$ (see Sect. 11.3).

It is appropriate again to point out that the deflection of the vertical is relatively more affected by the Molodensky correction than is the height anomaly. In extreme cases, this correction may attain values of a few seconds, as studies of models by Molodensky (Molodenski et al. 1962: pp. 217–225) indicate. This is considerable, since $1''$ in the deflection corresponds to 30 m in position. Numerical estimates will be found in Chap. 11.

We may summarize the result of applying Stokes' and Vening Meinesz' formulas to free-air anomalies directly, without any corrections. Stokes' formula yields height anomalies ζ with high accuracy; for many practical purposes, we may, in addition, identify these height anomalies with the corresponding geoidal undulations N. Vening Meinesz' formula gives deflections of the vertical at ground level that are relatively less accurate but often acceptable.

An advantage of the modern theory is its direct relation to the external gravity field of the earth, which is particularly important nowadays for the computation of the effect of gravitational disturbances on spacecraft trajectories and satellite orbits. It is immediately clear that ground-level quantities, such as free-air gravity anomalies, are better suited for this purpose than the corresponding quantities referred to the geoid, which is separated from the external field by the outer masses. For the computation of the external field and of spherical harmonics, the method described in Sect. 8.6.5

is particularly appropriate (see also Sect. 6.5).

Practically it is usually adequate to consider only the linear approximation by using (8–57). In many cases it is even possible to neglect the correction $-(\partial\Delta g/\partial h)\,h$, identifying the sea-level free-air anomalies $\Delta g^{\mathrm{harmonic}}$ with the corresponding ground-level anomalies Δg. In agreement with Sect. 3.9, these free-air anomalies $\Delta g^{\mathrm{harmonic}} = \Delta g$ may also be considered approximations to condensation anomalies in the sense of Helmert. This approximation is particularly sufficient for the external gravity field, spherical harmonics, and geoidal undulations or height anomalies. For deflections of the vertical, it is often necessary to use a more careful approach, such as the consideration of the indirect effect with mass-transporting gravity reductions (Sect. 8.2) or the modern methods of Sect. 8.9.

In high and steep mountains, the approach of Molodensky and others through free-air anomalies encounters practical difficulties, such as unreliability of interpolation, large corrections, and other computational problems. To avoid this, isostatic reduction in the modern sense shoud be used. Thus the clash between "conventional" (geoid) and "modern" (Molodensky-type) ideas gives way to an important synthesis. For another synthesis, see least-squares collocation in Sects. 10.2 and 11.2.

For further study, especially of the historic aspects, the reader is referred to the book by Molodenski et al. (1962) and the M.S. Molodensky Anniversary Volume edited by Moritz and Yurkina (2000).

Part II: Astrogeodetic methods according to Molodensky

8.12 Some background

The computation of a detailed geoid, or of a detailed gravity potential field, in limited areas, especially in mountainous regions, has not been very much in the focus of attention recently. There may be various reasons for this.

For decades now, global geoid determinations, either from satellite data or from a combination of satellite and gravimetric data have been in the center of interest (Lerch et al. 1979, Reigber et al. 1983, Rapp 1981). Even (almost) purely gravimetric global and local geoids have been successfully computed (March and Chang 1979), between the classic Heiskanen (1957) and the modern local geoid (Kühtreiber 2002 b). An excellent recent reference volume is that by Tsiavos (2002).

Over the oceans, the geoid is now known to an accuracy of perhaps a few centimeters, due to satellite altimetry. Unfortunately, satellite altimetry does not work over land areas. The classical method for a detailed geoid determination on the continents has been the gravimetric method, in spite of the fact that it is severely handicapped by lack of an adequate gravity coverage (or lack of information on such a coverage). Thus, we have the paradoxical situation that on the oceans, long a stepchild of geodesy, the geoid is now in general known much better than on the continents.

Still, the gravimetric method has continued to fascinate theoreticians because it gives rise to very interesting and deep mathematical problems, related to the geodetic boundary-value problem discussed above in this chapter.

These enormous practical and theoretical developments concerning global satellite and gravimetric gravity field determination have somewhat overshadowed the determination of detailed geoids in smaller areas, in particular, astrogeodetic geoids. Especially in mountainous regions, local geoid determinations are difficult. The gravimetric method does not work very well in high mountains. The astrogeodetic method, using astronomical observations of latitude and longitude, does work well there but is considered time-consuming and somewhat old-fashioned, perhaps also because working during the night is not very popular nowadays. An appropriate use of gravity and astrogeodetic data in high mountains must involve some topographic-isostatic reduction. Furthermore, the theory behind the astrogeodetic method is not nearly as attractively difficult as the theory of Molodensky's problem. Last but not least, high-mountain areas are exceptional and, apart from such countries as Switzerland and Austria, are frequently regions of little economic interest. For these and similar reasons, the mainstream of geodetic practice and theory has flown with grand indifference around high mountains, ignoring such trivial obstacles.

Still, a country such as Switzerland has made a virtue out of necessity and has traditionally been very active in local astrogeodetic geoid determination (Elmiger 1969, Gurtner 1978, Gurtner and Elmiger 1983). Austria has followed up (Österreichische Kommission für die Internationale Erdmessung 1983). It has been found that, even besides the problem of getting the required observations, the underlying theory is not so trivial as one might think and shows quite interesting features.

Concerning measurements, astronomical observations have again proved very feasible in mountains; see the articles by Erker, Bretterbauer and Gerstbach, Lichtenegger and Chesi in Chap. 2 of Österreichische Kommission für die Internationale Erdmessung (1983), followed by Sünkel et al. (1987). The main advantages of astrogeodetic versus gravimetric data for local geoid determination in mountainous regions may be summarized as follows:

1. It is sufficient to have astrogeodetic deflections of the vertical in the

region of geoid determination; no data are needed outside that region as they would be in the gravimetric method.

2. Errors in the topographic height have significantly less influence on deflections than on gravity data. Thus, a relatively crude terrain model will be sufficient for the use of astrogeodetic data.

As a matter of fact, the two types of data are not mutually exclusive; an optimal geoid determination will combine astrogeodetic deflections of the vertical, gravity anomalies, and possibly data of other type. A suitable technique for this purpose is least-squares collocation to be discussed in Chap. 10.

From the observational point of view it is interesting to note that *inertial surveying techniques* will be able to furnish deflections of the vertical and gravity anomalies rapidly and with sufficient accuracy for many purposes.

Let us finally try to give a list of various methods of geoid determination:

- conventional satellite techniques (Doppler, laser, etc.),
- satellite-to-satellite tracking,
- satellite gradiometry,
- satellite altimetry,
- gravimetry,
- astrogeodesy, and,
- most directly, GPS leveling (Sect. 4.6).

As a general rule, these methods are listed in such a way as to start with the most global and end up with the most local method, that is, according to decreasing globality or increasing locality. In general, going down the list also corresponds to increasing resolution and accuracy.

Again it should be stressed that these methods complement each other and should be combined for best results.

New satellite gravity missions have been discussed in Sect. 7.6.

Astrogeodetic method according to Molodensky

The remaining part of this chapter deals primarily with the lower end of the list, providing a detailed theory of astrogeodetic local geoid determination in areas with difficult topography. The role (and necessity) of topographic-isostatic reduction is investigated in detail. The computations for Austria give concrete numerical results for questions which have been much discussed theoretically, such as the difference between geoidal heights and height anomalies according to Molodensky (quasigeoidal heights), or the numerical effect of analytical continuation from the earth's surface to sea level (Österreichische Kommission für die Internationale Erdmessung 1983).

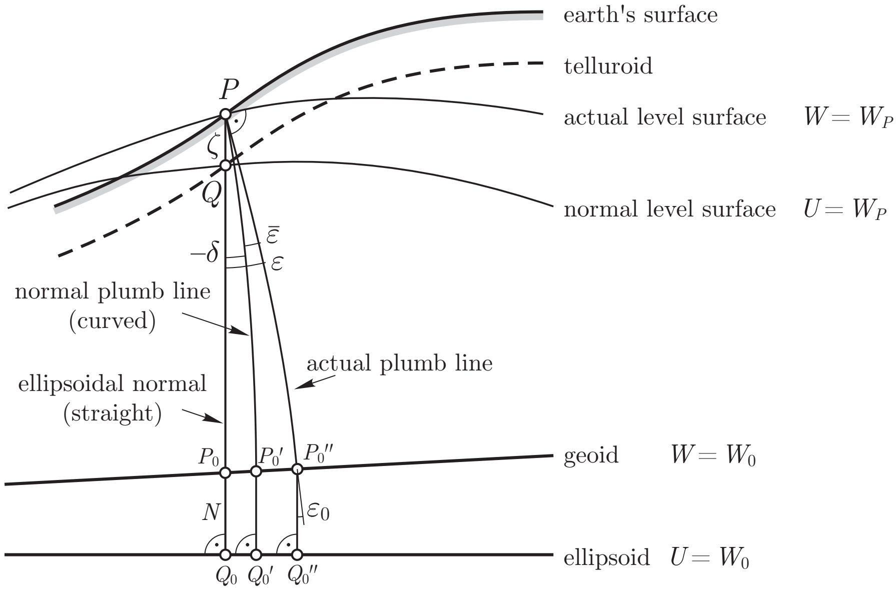

Fig. 8.13. The basic geometry

As a warm-up, let us return to basics and remember some main principles of Molodensky's geometry. Figure 8.13 illustrates the basic quantities. In the classical theory, the geoid is defined by its deviation N from a reference ellipsoid; N is the geoidal height. The geoid is a level surface $W = W_0 =$ constant of the gravity potential W; the ellipsoid is defined to be the level surface $U = U_0 =$ constant of a normal gravity potential U; the constants W_0 and U_0 are usually assumed to be equal (Sect. 2.12).

For the modern theory according to Molodensky (Sect. 8.4), to each point P of the earth's surface we associate a point Q in such a way that Q lies on the straight ellipsoidal normal through P and that

$$U(Q) = W(P) . \tag{8–121}$$

That is, Q is defined such that its normal potential U equals the actual potential W of P.

This corresponds to the classical relation

$$U_0 = U(Q_0) = W(P_0) = W_0 \tag{8–122}$$

mentioned above, by which U_0 is taken to be equal to W_0 (Fig. 8.13). By the same correspondence, the height anomaly according to Molodensky,

$$\zeta = Q\,P , \tag{8–123}$$

is the modern equivalent of the classical geoidal height,

$$N = Q_0 P_0 . \tag{8–124}$$

Using the anomalous potential

$$T = W - U , \tag{8–125}$$

we have according to Bruns' theorem

$$\zeta = \left(\frac{T}{\gamma}\right)_Q , \quad N = \left(\frac{T}{\gamma}\right)_{Q_0} , \tag{8–126}$$

where γ denotes the ellipsoidal normal gravity.

The points P_0 form the geoid, and the points Q_0 constitute the ellipsoid, both being level surfaces (of W and U, respectively). On the other hand, the points P form the earth's surface, and the set of points Q defines an auxiliary surface, denoted as telluroid according to R.A. Hirvonen. As a matter of fact, neither the earth's surface nor the telluroid are level surfaces, which makes matters more complicated than in the classical situation, where we deal with level surfaces.

Following a suggestion of Molodensky, one could plot the height anomalies ζ as vertical distances from the reference ellipsoid. Thus one obtains a geoid-like surface, the quasigeoid, and ζ could be considered as quasigeoidal heights. In contrast to the geoid, however, the quasigeoid is not a level surface and does not admit of a natural physical interpretation. Therefore, working with height anomalies ζ, it is best to consistently consider them quantities referred to the earth's surface (vertical distances between earth surface and telluroid), rather than using the quasigeoidal concept. A summary will be given in Sect. 8.15.

The classical gravity anomaly Δg_0 at sea level is defined as

$$\Delta g_0 = g(P_0) - \gamma(Q_0) , \tag{8–127}$$

where g denotes gravity and γ normal gravity. So far, $g(P_0)$ denotes the actual gravity on the geoid; we are not yet here considering mass-transporting gravity reductions.

Analogously we have according to Molodensky:

$$\Delta g = g(P) - \gamma(Q) . \tag{8–128}$$

Generally we will, as far as feasible, use the subscript "0" to designate quantities referred to sea level, to distinguish them from quantities referred to the earth's surface, which do not carry such a subscript. For instance, Δg_0

refers to sea level and Δg to the earth's surface. With GPS we have gravity disturbances

$$\Delta g = g(P) - \gamma(P) \,. \tag{8–129}$$

Regarding plumb line definition, we must distinguish three lines (Fig. 8.13):

1. the straight ellipsoidal normal $Q_0\,P$,
2. the actual plumb line $P_0''\,P$,
3. the normal plumb line $P_0'\,P$.

Geometrically, the ellipsoidal normal is defined as the straight line through P perpendicular to the ellipsoid. The (actual) plumb line is defined by the condition that, at each point of the line, the tangent coincides with the gravity vector $\mathbf{g}$ at that point; the plumb line is very slightly curved, but its curvature is irregular, being determined by the irregularities of topographic masses. The normal plumb line, at each of its points, is tangent to the normal gravity vector $\boldsymbol{\gamma}$; it possesses a curvature that is even smaller and completely regular.

The points P_0, P_0', and P_0'' coincide within a few decimeters, and we will not distinguish them in what follows. The reason is that the distance, in arc seconds, between P_0 and P_0'' is much smaller than the effect of plumb line curvature (Sect. 5.15). The same applies for Q_0, Q_0', and Q_0''.

The direction of the gravity vector $\mathbf{g}$ is the direction of (the tangent to) the plumb line. It is determined by two angles, the astronomical latitude Φ and the astronomical longitude Λ. Let Φ, Λ be referred to the earth's surface (to point P) and Φ_0, Λ_0 to the geoid (strictly speaking, to point P_0''). The differences

$$\delta\varphi = \Phi_0 - \Phi \,, \qquad \delta\lambda = \Lambda_0 - \Lambda \tag{8–130}$$

express the effect of plumb line curvature (Fig. 8.14). You may also wish to refer back to Fig. 5.18. Hence, we have

$$\Phi_0 = \Phi + \delta\varphi \,, \qquad \Lambda_0 = \Lambda + \delta\lambda \,. \tag{8–131}$$

Knowing the plumb line curvature $\delta\Phi, \delta\Lambda$, we could use these simple formulas to compute the sea-level values Φ_0, Λ_0 from the observed surface values Φ, Λ.

In the same way as Φ, Λ are related to the actual plumb line, the ellipsoidal latitude φ and the ellipsoidal longitude λ refer to the straight ellipsoidal normal. The quantities

$$\xi = \Phi - \varphi \,, \qquad \eta = (\Lambda - \lambda)\,\cos\varphi \tag{8–132}$$

are the components of the deflection of the vertical in a north-south and an east-west direction. For an arbitrary azimuth α, the vertical deflection ε is given by

$$\varepsilon = \xi\cos\alpha + \eta\,\sin\alpha \,. \tag{8–133}$$

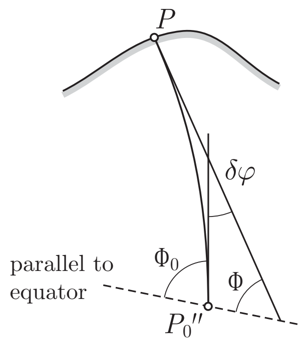

Fig. 8.14. Curvature of the plumb line along a north-south profile

These quantities ξ, η, ε refer to the earth's surface. Figure 8.13 shows ε.

Similarly, we have for the geoid

$$\xi_0 = \Phi_0 - \varphi\,, \qquad \eta_0 = (\Lambda_0 - \lambda)\cos\varphi\,, \tag{8–134}$$

$$\varepsilon_0 = \xi_0 \cos\alpha + \eta_0 \sin\alpha\,. \tag{8–135}$$

See again Fig. 8.13 for ε_0, noting that we do not distinguish the normals in Q_0 and Q_0'' as we have mentioned above.

In addition, we need the normal direction of the plumb line at the surface point P; it is defined as the tangent to the normal plumb line at P; the corresponding latitude and longitude will be denoted by $\bar{\varphi}, \bar{\lambda}$. In this "local" notation, there is no danger of confusion with the spherical coordinate $\bar{\varphi}$ used in earlier chapters. Hence, we have

$$\varphi = \bar{\varphi} + \delta\varphi_{\mathrm{normal}}\,, \qquad \lambda = \bar{\lambda} + \delta\lambda_{\mathrm{normal}}\,, \tag{8–136}$$

where $\delta\varphi$, $\delta\lambda$ express the normal plumb line curvature. These equations are the "normal equivalent" to (8–131): the "normal surface values" $\bar{\varphi}, \bar{\lambda}$ correspond to the "actual surface values" Φ, Λ and the ellipsoidal values φ, λ correspond to the geoidal values Φ_0, Λ_0. To make the analogy complete, we should replace $\varphi = \varphi(P_0)$ by $\varphi(P_0')$, but we have consistently neglected such differences.

In contrast to the actual plumb line curvature, it is very easy to compute the normal curvature of the plumb line: from (5–147) we have

$$\delta\varphi_{\mathrm{normal}} = -0.17''\, h_{[\mathrm{km}]} \sin 2\varphi\,, \qquad \delta\lambda_{\mathrm{normal}} = 0\,, \tag{8–137}$$

where $h_{[\mathrm{km}]}$ denotes elevation in kilometers.

Since the ellipsoidal normal and hence φ, λ are geometrically defined, we may call the quantities (8–132) "geometric deflections of the vertical" at the earth's surface. On the other hand, the normal plumb line is physically (or dynamically) defined by means of the external gravity field of an equipotential ellipsoid. Hence also $\bar{\varphi}, \bar{\lambda}$ are dynamically defined. The quantities obtained by replacing φ, λ by $\bar{\varphi}, \bar{\lambda}$ so that

$$\bar{\xi} = \Phi - \bar{\varphi}\,, \qquad \bar{\eta} = (\Lambda - \bar{\lambda})\cos\varphi\,, \tag{8–138}$$

are called "dynamical deflections of the vertical" at the earth's surface. By (8–136) and (8–137) we have

$$\bar{\xi} = \xi + \delta\varphi_{\text{normal}}\,, \qquad \bar{\eta} = \eta\,, \tag{8–139}$$

since $\delta\lambda_{\text{normal}} = 0$. For an azimuth α we accordingly have

$$\bar{\varepsilon} = \bar{\xi}\cos\alpha + \bar{\eta}\sin\alpha\,. \tag{8–140}$$

Compare ε and $\bar{\varepsilon}$ in Fig. 8.13 and note that in this figure δ denotes the curvature of the normal plumb line for the azimuth α given by the analogous formula

$$\delta = \delta\varphi_{\text{normal}}\cos\alpha + (\delta\lambda_{\text{normal}}\cos\varphi)\sin\alpha = \delta\varphi_{\text{normal}}\cos\alpha\,. \tag{8–141}$$

8.13 Astronomical leveling revisited

From Fig. 8.15 we take the well-known differential relation

$$dN = -\varepsilon_0\, ds\,, \tag{8–142}$$

where ε_0 denotes the deflection of the vertical at the geoid. Integration between two points A and B yields the difference between their geoidal heights:

$$N_B - N_A = -\int_A^B \varepsilon_0\, ds\,, \tag{8–143}$$

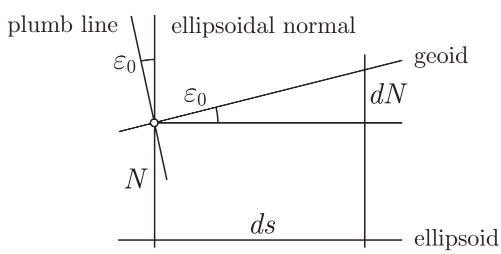

Fig. 8.15. Astronomical leveling according to Helmert

or, approximately,

$$N_B - N_A = -\frac{\varepsilon_{0A} + \varepsilon_{0B}}{2}\, s_{AB}\,, \tag{8–144}$$

where s_{AB} denotes the horizontal distance between A and B. The minus sign is conventional. Cf. Sect. 5.14.

A corresponding relation to height anomalies according to Molodensky is found as follows (Molodensky et al. 1962: p. 125):

$$d\zeta = \frac{\partial \zeta}{\partial s}\, ds + \frac{\partial \zeta}{\partial h}\, dh\,, \tag{8–145}$$

notations following Fig. 8.16. Since the earth's surface is not a level surface, we also have a vertical part $(\partial\zeta/\partial h)\, h$ in addition to the usual horizontal part $(\partial\zeta/\partial s)\, ds$. The vertical part arises from change in height and is usually smaller than the horizontal part.

In analogy to (8–142), the horizontal part is given by

$$\frac{\partial \zeta}{\partial s} = -\bar{\varepsilon}\,, \tag{8–146}$$

where $\bar{\varepsilon}$ denotes the dynamical deflection of the vertical at the earth's surface; cf. (8–140) and Fig. 8.13. For the vertical part we have from (8–126):

$$\frac{\partial \zeta}{\partial h} = \frac{\partial}{\partial h}\left(\frac{T}{\gamma}\right) = \frac{1}{\gamma}\left(\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h}\, T\right) \tag{8–147}$$

or

$$\frac{\partial \zeta}{\partial h} = -\frac{\Delta g}{\gamma} = -\frac{g-\gamma}{\gamma} \tag{8–148}$$

according to the fundamental equation of physical geodesy (8–36).

Hence (8–145) becomes

$$d\zeta = -\bar{\varepsilon}\, ds - \frac{g-\gamma}{\gamma}\, dh\,. \tag{8–149}$$

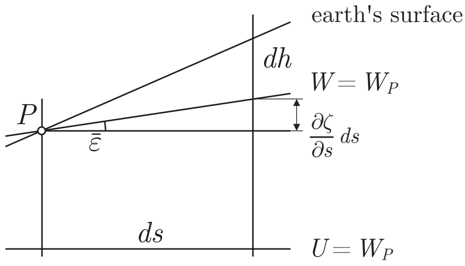

Fig. 8.16. Astronomical leveling according to Molodensky

Integrating this relation yields the difference of the height anomaly

$$\zeta_B - \zeta_A = -\int_A^B \bar{\varepsilon}\, ds - \int_A^B \frac{\Delta g}{\gamma}\, dh\,; \tag{8–150}$$

the gravity anomaly Δg refers to the earth's surface according to (8–128). The first term on the right-hand side represents the Helmert integral (8–143) of the surface deflection $\bar{\varepsilon}$, and the second term is Molodensky's correction to the Helmert integral, necessary to obtain height anomalies. This correction depends on the gravity g at the earth's surface.

8.14 Topographic-isostatic reduction of vertical deflections

For the reasons mentioned at the end of the preceding section, it is natural to try and find a way which makes use of the clear advantages of the topographic-isostatic reduction but avoids the problems inherent in a free-air reduction from the surface point P to the geoidal point P_0.

In Sect. 8.9, we have treated the reduction of gravity from the modern point of view. The second formula of (8–95) is

$$g^{\mathrm{c}} = g - \delta g\,. \tag{8–151}$$

Everything is referred to the ground point P, and $\delta g = \delta g_{\mathrm{TI}}$ is the effect of gravity reduction on g, also at P. In the *topographic-isostatic reduction which we use here exclusively*, it is the gravitational attraction of the topography minus the gravitational attraction of the compensating isostatic masses, topography minus isostasy.

To get the topographic-isostatic gravity anomaly, we subtract normal gravity γ, also referred to ground level, more precisely, to the corresponding telluroid point Q. Thus,

$$\Delta g^{\mathrm{c}} = \Delta g - \delta g_{\mathrm{TI}}. \tag{8–152}$$

The explanation is trivial: you are standing at point P and watch how the topography is removed to fill the isostatic mass deficits, but by a miracle you are still hovering at P, now in "free air".

Application to deflections of the vertical

The gravity anomaly is only one component of the anomalous gravity vector, the other two being the vertical components ξ and η, both, of course, multiplied by γ to get the dimensions right. *Thus, ξ and η can be isostatically reduced in exactly the same way.*

For ξ and η, (8–152) becomes

$$\xi^{c} = \xi - \xi_{TI} + \delta\varphi_{normal}\,, \qquad \eta^{c} = \eta - \eta_{TI}\,. \tag{8–153}$$

By means of (8–139) this may be written

$$\xi^{c} = \bar{\xi} - \xi_{TI}\,, \qquad \eta^{c} = \bar{\eta} - \eta_{TI}\,. \tag{8–154}$$

The interpretation of (8–154), however, is clear, simple, and rigorous: from the dynamic deflections of the vertical at P, which are the very quantities $\bar{\xi}$ and $\bar{\eta}$, we subtract the effect of the topographic-isostatic masses, ξ_{TI} and η_{TI} likewise at P. The vertical deflections so obtained, ξ^{c} and η^{c}, thus do not really refer to the (co-)geoid; in reality, *they refer to the earth's surface!*

But what, then, means the normal plumb line curvature $\delta\varphi_{normal}$ in (8–153)? Does it not mean a reduction from the earth's surface to sea level? No, in Eqs. (8–139) it only denotes the transformation between the geometrical and the dynamical deflection of the vertical, both referred to the point P of the earth's surface. This is also clear from Fig. 8.13, which illustrates the formula

$$\bar{\varepsilon} = \varepsilon + \delta\,, \tag{8–155}$$

extending (8–139) to an arbitrary azimuth, δ being defined by (8–141).

This interpretation of (8–153) or (8–154) as isostatically reduced deflections of the vertical at the earth's surface is exact, whereas the interpretation of (8–8) as deflections at the cogeoid was only approximate. This is the desired rigorous interpretation of our isostatically reduced vertical deflections.

This interpretation exactly corresponds to the modern view of gravity reduction according to the theory of Molodensky. According to this view, the isostatically (or in some other way) reduced gravity anomalies continue to refer to the earth's surface. The classical gravity reduction (Sect. 8.2) had comprised two procedures: mass transport and shift $P \rightarrow P_0$; the new view of gravity reduction only considers the mass transport; the problematic shift $P \rightarrow P_0$ is avoided.

Formally, a "normal free-air reduction"

$$F = -\frac{\partial\gamma}{\partial h}\,h \tag{8–156}$$

may be said to occur also in Molodensky's theory: normal gravity γ in the new definition (8–128) of the gravity anomaly, where it refers to the telluroid point Q, is computed by

$$\gamma = \gamma_{Q_0} + \frac{\partial\gamma}{\partial h}\,h\,, \tag{8–157}$$

with $h = Q_0\,Q$ denoting the normal height of P. But instead of reducing actual gravity g downward, from P to P_0, now normal gravity is reduced upward from Q_0 to Q. Whereas for the first process the use of the normal gradient $\partial\gamma/\partial h$ is problematic, it is fully justified for the second process.

In a similar way, we might interpret $\delta\varphi_{\text{normal}}$ as a reduction of φ for normal curvature of the plumb line upwards, say, from P_0 to P. This is possible because in (8–136) φ could be said to refer to P_0' (because P_0 and P_0' practically coincide), and because $\bar{\varphi}$ denotes the latitude of the tangent to the normal plumb line at P. This interpretation is instructive because of the analogy with gravity reduction, though regarding φ and $\bar{\varphi}$ as ellipsoidal and dynamic latitude of the same point P appears more natural. Refer again to our key figure (Fig. 8.13).

As pointed out above, the present interpretation of ξ^{c} and η^{c} as isostatically reduced deflections of the vertical at the earth's surface is conceptually rigorous and therefore also practically more accurate, but this decisive advantage implies a computational drawback if integration along a profile is used: Since this integration must now be performed along the earth's surface and not along a level surface such as the geoid, computation will be more complicated. Instead of the simple Helmert formula (8–143), we now must use the Molodensky formula (8–150):

$$\zeta_B^{\mathrm{c}} - \zeta_A^{\mathrm{c}} = -\int_A^B \varepsilon^{\mathrm{c}}\, ds - \int_A^B \frac{g^{\mathrm{c}} - \gamma}{\gamma}\, dh \tag{8–158}$$

with

$$\varepsilon^{\mathrm{c}} = \xi^{\mathrm{c}} \cos\alpha + \eta^{\mathrm{c}} \sin\alpha\,, \tag{8–159}$$

and $\Delta g^{\mathrm{c}} = g^{\mathrm{c}} - \gamma$, where g^{c} is the isostatically reduced surface value of gravity (measured value g minus attraction of the topographic-isostatic masses).

From the isostatic height anomalies ξ^{c} obtained in this way, we then get the actual height anomalies ζ by applying the indirect effect:

$$\zeta = \zeta^{\mathrm{c}} + \delta\zeta \tag{8–160}$$

with

$$\delta\zeta = \frac{T_{\mathrm{TI}}}{\gamma}\,. \tag{8–161}$$

This is completely analogous to (8–5) and (8–3), but now T_{TI} is the potential of the topographic-isostatic masses at the surface point P. As a matter of fact, normal gravity in (8–3) refers to the ellipsoid, and in (8–161) to the telluroid, but the difference is generally small.

For higher mountains, the isostatic reduction procedure described in the present section is preferable in practice to a direct application of Molodensky's formula (8–150) because the isostatically reduced vertical deflections are much smoother and easier to interpolate. It is, however, extremely laborious from a computational point of view since the integration must be performed along the earth's surface (or, what is practically the same, along the telluroid).

We remark that the computational drawback of the present method, the Molodensky integration along the earth's surface, can be completely avoided if we perform our computations in space: instead of integrating along a surface, we perform collocation in space. This modern procedure, to be described in the next chapter, permits a simple and computationally convenient use of surface deflections and also their combination with gravimetric and other data. Still, the present developments are necessary for a full understanding of the collocation approach.

Final remarks

In these last sections we tried to apply the same principle for topographic-isostatic reduction (the "remove-restore method") *at point level* to all terrestrial data related to the gravity vector: gravity anomalies and disturbances (Sect. 8.9) and deflections of the vertical (Sect. 8.14). This unified view of isostatically reduced data thus makes them directly suitable for combined solutions by least-squares collocation to be treated in Chap. 10.

8.15 The meaning of the geoid

We now review the geoid and some surfaces that might be able to replace it. We will again confirm the unique role of the geoid as a standard surface of physical geodesy.

The meaning of the geoid is very simple. It is defined in Sect. 2.2 as one of the equipotential surfaces (level surfaces, surfaces of constant gravity potential)

$$W(x, y, z) = \text{constant}. \tag{8–162}$$

The constant is chosen so that, on the oceans, the geoid coincides with mean sea level:

$$W(x, y, z) = W_0. \tag{8–163}$$

This is the usual classical equation of the geoid. So what is the problem? Well, theory and practice are different, in geodesy as well as in daily life. First, we must disregard small tidal effects (on the order of 50 cm). This is done by applying a suitable tidal model and is not too problematic. In fact, we have numerous geoids determined from satellite observations. Second, they are usually expressed in terms of a series of spherical harmonics. If taken at sea level, such a series may diverge (this is related to the difficulties of downward continuation, cf. Sect. 8.6). Such a possible divergence may concern mathematicians, but it should not concern geodesists, for several reasons:

1. Our spherical-harmonic expansions are not infinite series but finite polynomials, by their very determination and computations. So divergence problems do not exist; the question is only good approximation.

2. Such approximating polynomials of spherical harmonics always exist for arbitrary accuracy requirements (Frank and Mises 1930: p. 760). In geodesy we usually speak of Runge's theorem. The whole subject is thorougly discussed in Moritz (1980 a: Sects. 6 to 8).

3. If you use spatial collocation, the behavior (harmonic or not, convergent or divergent, ...) of the solution is completely determined by the covariance function used. One always uses "good" covariance functions, which are harmonic and analytic down to a sphere completely inside the earth.

So forget all about the convergence problem. *It is practically solved.* Further discussions beyond the results obtained so far would have to be made at a very high mathematical level. The question can be made as complicated as desired; if looked at it from the right angle, it is simple.

Geoid and downward continuation

Therefore, and by the reasoning at the end of Sect. 10.1, the geoid computed by (harmonic!) spherical-harmonic expressions and by collocation is not a level surface of the actual geopotential W but a *level surface of a harmonic downward continuation* of W, for the simple reason that the base functions both of spherical harmonics and of collocation satisfy Laplace's equation (8–2). We may speak of a "harmonic geoid". This again emphasizes the importance of analytical continuation (Sect. 8.6). We have deliberately used the indefinite article "a" in the italicized expression above, because harmonic downward continuation is an inverse problem and thus has no unique solution (see below).

The application of collocation to ξ, η, Δg *without gravity reduction* gives height anomalies ζ and undulations of the harmonic geoid, N^{harmonic}, by simply varying the elevation parameter (h and zero, respectively) in the collocation program. A completely analogous fact was remarked at the end of the last section for the case of height anomalies ζ^{c} and cogeoidal heights N^{c}. In the case of Molodensky's problem (without or with gravity reduction), we have seen a completely similar behavior with the application of the generalized Stokes and Vening Meinesz formulas, (8–75) and (8–76).

Geoid, harmonic geoids, and quasigeoid

The *geoid in the usual sense* of Eqs. (2–18) or (8–163) is defined purely by nature and is independent of geodetic observations (except for the tidal corrections). Its disadvantage is that it depends on the "topographic masses" above the geoid whose density is unknown, at least in principle. This drawback seems to be theoretical rather than practical.

The *harmonic geoids* are equipotential surfaces of an analytical downward continuation. We shall be careful to denote the harmonically continued potential by W^{harmonic} so that

$$W^{\text{harmonic}} = W_0 = \text{constant} \tag{8–164}$$

denote harmonic geoid(s).

To repeat, analytical downward continuation based on discrete data at the earth's surface is an *inverse problem* (Sect. 1.13; for more information see www.inas.tugraz.at/forschung/InverseProblems/AngerMoritz.html) which has infinitely many possible solutions. For collocation, e.g., each solution corresponds to the choice of a different covariance function.

Thus, the "harmonic geoid" is not uniquely defined. It is a product not only of nature but also of the computational method used. It cannot, therefore, replace the real geoid as a standard surface.

The "cogeoids" of the various gravity reductions (Sect. 8.2) are intermediate computational concepts and should never be used in place of the geoid. The topographic-isostatic height anomalies at point level, ζ^{c}, and the heights of the topographic-isostatic cogeoid, N^{c}, are related to each other by *analytical continuation*. The same collocation formula applies if the height anomaly $f(P)$ is computed at sea level with elevation parameter 0 to give N^{c}, or, if $f(P)$ is computed at point level with elevation parameter h, to give ζ^{c}. (The elevation parameter h is a height above sea level in any of the definitions of Chap. 4.) See item 5 at the end of Sect. 10.2.

For the limiting case of Fig. 8.5 c, take the question: "How is the undulation N^{harmonic} of a 'harmonic geoid' related to the height anomaly ζ above it on the ground and on the same vertical?" Answer: "By analytic continuation!"

Another special question to which the answer is also easy: "Which gravity reduction leaves the geoid unchanged?" Answer: "The Rudzki reduction" (Sect. 3.8). So why not use it? It changes the external potential, which today is of paramount importance.

"What is the difference between the Rudzki reduction and the harmonic downward continuation?" Answer: "The Rudzki reduction leaves the geoid unchanged but changes the external geopotential: there is $W^{\text{c}} = W = W_0$ *only on the geoid,* but $W^{\text{c}} \neq W$ outside the earth, which is inadmissible.

The harmonic continuation leaves the external geopotential unchanged but changes the geoid: $W^{\text{harmonic}} = W$ outside the earth and on the earth's surface, but $W^{\text{harmonic}} \neq W$ at sea level.

Height anomalies and quasigeoid

The height anomalies ζ refer to the physical earth's surface. They find their natural physical interpretation in Hirvonen's *telluroid.* Molodensky proposed to plot ζ above the reference ellipsoid and get the "quasigeoid". Thus, ζ gives the quasigeoid in exactly the same way as the geoidal height N gives the geoid. However, this analogy is purely formal. There is no way to interpret the quasigeoid as a surface of constant potential or find any other physical interpretation for it. Again, it cannot replace the real geoid as a standard surface.

Thus, in spite of all modern developments, the geoid retains its role as a standard reference for physical geodesy. However, the reader must have a clear view of all the concepts reviewed in this section, see Forsberg and Tscherning (1997).

A final remark on the many facets of free-air reduction

Now, dear reader, having struggled through almost the whole book, you will be able to understand the *disjecta membra* on free-air reduction strewn all over it, such as Sects. 3.3, 3.9, 8.2, 8.6, 8.9, and the present section. Have a couple of nice mountaineering tours!