6 Gravity field outside the earth

6.1 Introduction

The gravity field outside the earth is particularly important at satellite altitude; this will be treated mainly in Chap. 7. The considerations of the present chapter are applicable to gravitational forces also at satellites (see Sect. 7.2), but their main practical purpose is to compute test values for the gravity vector, gravity disturbances, and gravity anomalies at flight elevations for comparison with airborne gravimetry for reference and calibration purposes. Airborne gravimetry is much faster than both terrestrial and shipborne gravimetry, so it is of interest also for geophysical prospecting.

For computational reasons, it is again convenient to split the gravity potential W and the gravity vector

$$\mathbf{g} = \text{grad } W \tag{6–1}$$

into a normal potential U and a normal gravity vector

$$\boldsymbol{\gamma} = \text{grad } U \,, \tag{6–2}$$

and the disturbing potential $T = W - U$ and the gravity disturbance vector

$$\delta\mathbf{g} = \text{grad } T = \mathbf{g} - \boldsymbol{\gamma} \,. \tag{6–3}$$

The normal gravity field is usually taken to be the gravity field of a suitable equipotential ellipsoid. This permits closed formulas and offers other advantages of mathematical simplicity (see Sect. 2.12).

Thus, U and $\boldsymbol{\gamma}$ are computed first, and W and $\mathbf{g}$ are then obtained by

$$\begin{aligned} W &= U + T \,, \\ \mathbf{g} &= \boldsymbol{\gamma} + \delta\mathbf{g} \,. \end{aligned} \tag{6–4}$$

For some purposes, we need the vector of gravitation, grad V (pure attraction without centrifugal force), rather than the vector of gravity. The gravitational vector is computed from the gravity vector by subtracting the vector of centrifugal force:

$$\text{grad } V = \mathbf{g} - \text{grad } \Phi = \mathbf{g} - \begin{bmatrix} \omega^2 x \\ \omega^2 y \\ 0 \end{bmatrix} , \tag{6–5}$$

where the notations of Sect. 2.1 are used. The rectangular coordinate system x, y, z will be applied in this chapter in the usual sense: it is geocentric, the x- and y-axes lying in the equatorial plane with Greenwich longitudes 0° and 90° East, respectively, and the z-axis being the rotation axis of the earth.

The sign of the components of $\mathbf{g}$, $\boldsymbol{\gamma}$, $\delta\mathbf{g}$, etc., will always be chosen so that they are positive in the direction of increasing coordinates.

6.2 Normal gravity vector

The gravity field of an equipotential ellipsoid is best expressed in terms of ellipsoidal-harmonic coordinates u, β, λ, introduced in Sects. 1.15 and 2.7. They are related to rectangular coordinates x, y, z by

$$\begin{aligned} x &= \sqrt{u^2+E^2}\,\cos\beta\,\cos\lambda\,, \\ y &= \sqrt{u^2+E^2}\,\cos\beta\,\sin\lambda\,, \\ z &= u\,\sin\beta\,. \end{aligned} \tag{6–6}$$

If x, y, z are given, then u, β, λ can be computed by closed formulas. First we find

$$x^2+y^2=(u^2+E^2)\cos^2\beta\,,\qquad z^2=u^2\sin^2\beta\,. \tag{6–7}$$

Eliminating β between these two equations, we obtain a quadratic equation for u^2, whose solution is

$$u^2=(x^2+y^2+z^2-E^2)\left[\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4E^2z^2}{(x^2+y^2+z^2-E^2)^2}}\,\right]. \tag{6–8}$$

Then β is given by

$$\tan\beta=\frac{z\sqrt{u^2+E^2}}{u\sqrt{x^2+y^2}}\,, \tag{6–9}$$

and for λ we simply have

$$\tan\lambda=\frac{y}{x}\,. \tag{6–10}$$

With known ellipsoidal-harmonic coordinates, the normal potential U is given by (2–126):

$$U(u,\,\beta)=\frac{GM}{E}\,\tan^{-1}\frac{E}{u}+\tfrac{1}{2}\,\omega^2a^2\frac{q}{q_0}\left(\sin^2\beta-\tfrac{1}{3}\right)+\tfrac{1}{2}\,\omega^2(u^2+E^2)\,\cos^2\beta\,. \tag{6–11}$$

The components of $\boldsymbol{\gamma}$ along the coordinate lines are, by (2–131) and (2–132),

$$
\begin{aligned}
\gamma_u &= \frac{1}{w}\frac{\partial U}{\partial u} = -\frac{1}{w}\left[\frac{GM}{u^2+E^2} + \frac{\omega^2 a^2 E}{u^2+E^2}\frac{q'}{q_0}\left(\tfrac{1}{2}\sin^2\beta - \tfrac{1}{6}\right) - \omega^2 u \cos^2\beta\right],\\
\gamma_\beta &= \frac{1}{w\sqrt{u^2+E^2}}\frac{\partial U}{\partial \beta} = -\frac{1}{w}\left[-\frac{\omega^2 a^2}{\sqrt{u^2+E^2}}\frac{q}{q_0} + \omega^2\sqrt{u^2+E^2}\right]\sin\beta\cos\beta\,,\\
\gamma_\lambda &= \frac{1}{\sqrt{u^2+E^2}\cos\beta}\frac{\partial U}{\partial \lambda} = 0\,.
\end{aligned}
\tag{6–12}
$$

To get the components of $\boldsymbol{\gamma}$ in the xyz-system, we compute

$$
\frac{\partial U}{\partial u} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial u}\,, \qquad \text{etc.} \tag{6–13}
$$

The partial derivatives of x, y, z with respect to u, β, λ are obtained by differentiating equations (6–6); we find

$$
\begin{aligned}
\frac{\partial U}{\partial u} &= \frac{u}{\sqrt{u^2+E^2}}\cos\beta\cos\lambda\frac{\partial U}{\partial x} + \frac{u}{\sqrt{u^2+E^2}}\cos\beta\sin\lambda\frac{\partial U}{\partial y} + \sin\beta\frac{\partial U}{\partial z}\,,\\
\frac{\partial U}{\partial \beta} &= -\sqrt{u^2+E^2}\sin\beta\cos\lambda\frac{\partial U}{\partial x} - \sqrt{u^2+E^2}\sin\beta\sin\lambda\frac{\partial U}{\partial y} + u\cos\beta\frac{\partial U}{\partial z}\,,\\
\frac{\partial U}{\partial \lambda} &= -\sqrt{u^2+E^2}\cos\beta\sin\lambda\frac{\partial U}{\partial x} + \sqrt{u^2+E^2}\cos\beta\cos\lambda\frac{\partial U}{\partial y}\,.
\end{aligned}
\tag{6–14}
$$

Introducing the components

$$
\gamma_x = \frac{\partial U}{\partial x}\,, \;\cdots\,; \qquad \gamma_u = \frac{1}{w}\frac{\partial U}{\partial u}\,, \;\cdots\,, \tag{6–15}
$$

we obtain

$$
\begin{aligned}
\gamma_u &= \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\cos\lambda\,\gamma_x + \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\sin\lambda\,\gamma_y + \frac{1}{w}\sin\beta\,\gamma_z\,,\\
\gamma_\beta &= -\frac{1}{w}\sin\beta\cos\lambda\,\gamma_x - \frac{1}{w}\sin\beta\sin\lambda\,\gamma_y + \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\,\gamma_z\,,\\
\gamma_\lambda &= -\sin\lambda\,\gamma_x + \cos\lambda\,\gamma_y\,.
\end{aligned}
\tag{6–16}
$$

These are the formulas of an orthogonal rectangular coordinate transformation. The inverse transformation is obtained by interchanging the rows and columns in the matrix of this equation system. Thus, we obtain

$$\begin{aligned}\gamma_x &= \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\,\cos\lambda\,\gamma_u - \frac{1}{w}\sin\beta\,\cos\lambda\,\gamma_\beta - \sin\lambda\,\gamma_\lambda\,,\\ \gamma_y &= \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\,\sin\lambda\,\gamma_u - \frac{1}{w}\sin\beta\,\sin\lambda\,\gamma_\beta + \cos\lambda\,\gamma_\lambda\,,\\ \gamma_z &= \frac{1}{w}\sin\beta\,\gamma_u + \frac{u}{w\sqrt{u^2+E^2}}\cos\beta\,\gamma_\beta\,.\end{aligned} \tag{6–17}$$

This follows from the definition of these coefficients as direction cosines. Equations (6–17) may also be found by solving the linear Eqs. (6–16) with respect to γ_x, γ_y, γ_z in some other way.

The formulas of the present section are completely rigorous. They can easily be programmed. Here it would not be appropriate to use the spherical approximation because they are relatively large quantities of the normal ellipsoidal field.

6.3 Gravity disturbance vector from gravity anomalies

In Sect. 1.4, we have introduced spherical coordinates: r (radius vector), ϑ (polar distance), λ (geocentric longitude) (see Fig. 1.3). Now we use these coordinates again but replace the polar distance ϑ by its complement, the geocentric latitude $\bar{\varphi}$ (Fig. 6.3). In analogy to (1–26), these spherical coor-

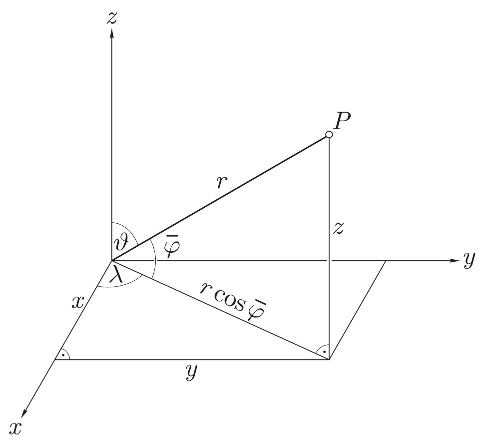

Fig. 6.1. Spherical coordinates $r, \bar{\varphi}$ (or ϑ, respectively), λ and rectangular coordinates x, y, z

dinates are related to rectangular coordinates x, y, z by the equations

$$\begin{aligned} x &= r \cos \bar{\varphi} \cos \lambda \,, \\ y &= r \cos \bar{\varphi} \sin \lambda \,, \\ z &= r \sin \bar{\varphi} \end{aligned} \tag{6–18}$$

or inversely by

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \,, \\ \bar{\varphi} &= \tan^{-1} \frac{z}{\sqrt{x^2 + y^2}} \,, \\ \lambda &= \tan^{-1} \frac{y}{x} \,. \end{aligned} \tag{6–19}$$

Now it is convenient to start with the components δg_r, $\delta g_{\bar{\varphi}}$, δg_λ of the gravity disturbance vector $\delta\mathbf{g}$, Eq. (6–3), in the spherical coordinates r, $\bar{\varphi}$, λ. In analogy to (2–377), we have

$$\delta g_r = \frac{\partial T}{\partial r} \,, \qquad \delta g_{\bar{\varphi}} = \frac{1}{r} \frac{\partial T}{\partial \bar{\varphi}} \,, \qquad \delta g_\lambda = \frac{1}{r \, \cos \bar{\varphi}} \frac{\partial T}{\partial \lambda} \,. \tag{6–20}$$

Since we are dealing with the relatively small quantities of the disturbing field, a spherical approximation may be sufficient (Sect. 2.13), as it was in the case of Stokes' formula.

The disturbing potential T may be expressed in terms of the free-air anomalies at the earth's surface by the formula of Pizzetti, Eqs. (2–302) and (2–303),

$$T_P = T(r, \bar{\varphi}, \lambda) = \frac{R}{4\pi} \iint_\sigma \Delta g \; S(r, \psi) \; d\sigma \,, \tag{6–21}$$

where $S(r, \psi)$ is the extended Stokes function,

$$S(r, \psi) = \frac{2R}{l} + \frac{R}{r} - 3 \frac{R\, l}{r^2} - \frac{R^2}{r^2} \cos \psi \left(5 + 3 \ln \frac{r - R \, \cos \psi + l}{2r} \right), \tag{6–22}$$

and

$$l = \sqrt{r^2 + R^2 - 2R\, r \cos \psi} \,. \tag{6–23}$$

According to (6–20), we must differentiate (6–21) with respect to r, $\bar{\varphi}$, and λ. Here we note that the integral on the right-hand side of (6–21) depends on r, $\bar{\varphi}$, λ only through the function $S(r, \psi)$. Thus, Δg being constant with respect to the differentiation, we have

$$\begin{aligned} \delta g_r &= \frac{R}{4\pi} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial r} \, d\sigma \,, \\ \delta g_{\bar\varphi} &= \frac{R}{4\pi\, r} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial \bar\varphi} \, d\sigma \,, \\ \delta g_\lambda &= \frac{R}{4\pi\, r \, \cos\bar\varphi} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial \lambda} \, d\sigma \,. \end{aligned} \tag{6–24}$$

The point P at which $\delta\mathbf{g}$ is to be computed has the coordinates $\bar\varphi$, λ; let the corresponding coordinates of the variable point P', to which Δg and $d\sigma$ refer, be denoted by $\bar\varphi'$, λ'. Then $d\sigma$ will be expressed by

$$d\sigma = \cos\bar\varphi' \, d\bar\varphi' \, d\lambda' \tag{6–25}$$

and ψ, the angular distance between P and P', is represented via

$$\cos\psi = \sin\bar\varphi \, \sin\bar\varphi' + \cos\bar\varphi \, \cos\bar\varphi' \cos(\lambda' - \lambda) \,. \tag{6–26}$$

We have

$$\frac{\partial S(r,\psi)}{\partial\bar\varphi} = \frac{\partial S(r,\psi)}{\partial\psi} \, \frac{\partial\psi}{\partial\bar\varphi} \,, \qquad \frac{\partial S(r,\psi)}{\partial\lambda} = \frac{\partial S(r,\psi)}{\partial\psi} \, \frac{\partial\psi}{\partial\lambda} \,. \tag{6–27}$$

Now we recall the corresponding derivations in Sect. 2.19, leading to Vening Meinesz' formula. As a spherical approximation which is sufficient for T, $\delta\mathbf{g}$, etc., we may identify the geocentric latitude $\bar\varphi$ with the ellipsoidal latitude φ. Thus, Eqs. (6–27) and (2–380) are completely analogous, and (2–383) may be borrowed from Sect. 2.19:

$$\frac{\partial\psi}{\partial\bar\varphi} = -\cos\alpha \,, \qquad \frac{\partial\psi}{\partial\lambda} = -\cos\bar\varphi \, \sin\alpha \,. \tag{6–28}$$

The azimuth α is given by formula (2–388):

$$\tan\alpha = \frac{\cos\bar\varphi' \sin(\lambda' - \lambda)}{\cos\bar\varphi \, \sin\bar\varphi' - \sin\bar\varphi \, \cos\bar\varphi' \cos(\lambda' - \lambda)} \,. \tag{6–29}$$

By means of (6–27) and (6–28), Eqs. (6–24) become

$$\begin{aligned} \delta g_r &= \frac{R}{4\pi} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial r} \, d\sigma \,, \\ \delta g_{\bar\varphi} &= -\frac{R}{4\pi\, r} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial\psi} \, \cos\alpha \, d\sigma \,, \\ \delta g_\lambda &= -\frac{R}{4\pi\, r} \iint_\sigma \Delta g \, \frac{\partial S(r,\psi)}{\partial\psi} \, \sin\alpha \, d\sigma \,. \end{aligned} \tag{6–30}$$

Now we form the derivatives of the extended Stokes function (6–22) with respect to r and ψ. By differentiating (6–23), we get

$$\frac{\partial l}{\partial r} = \frac{r - R\cos\psi}{l}\,, \qquad \frac{\partial l}{\partial \psi} = \frac{R\,r}{l}\sin\psi\,. \tag{6–31}$$

By means of these auxiliary relations, we find

$$\begin{aligned}
\frac{\partial S}{\partial r} &= -\frac{R\,(r^2 - R^2)}{r\,l^3} - \frac{4R}{r\,l} - \frac{R}{r^2} + \frac{6R\,l}{r^3}\\
&\quad + \frac{R^2}{r^3}\cos\psi\left(13 + 6\ln\frac{r - R\cos\psi + l}{2r}\right),\\
\frac{\partial S}{\partial \psi} &= \sin\psi\left[-\frac{2R^2 r}{l^3} - \frac{6R^2}{r\,l} + \frac{8R^2}{r^2}\right.\\
&\quad \left. + \frac{3R^2}{r^2}\left(\frac{r - R\cos\psi - l}{l\sin^2\psi} + \ln\frac{r - R\cos\psi + l}{2r}\right)\right].
\end{aligned} \tag{6–32}$$

Somewhat more convenient expressions are obtained by substituting

$$t = \frac{R}{r}\,, \tag{6–33}$$

$$D = \frac{l}{r} = \sqrt{1 - 2t\cos\psi + t^2}\,. \tag{6–34}$$

Then the extended Stokes function (6–22) and its derivatives (6–32) become

$$S(r,\psi) = t\left[\frac{2}{D} + 1 - 3D - t\cos\psi\left(5 + 3\ln\frac{1 - t\cos\psi + D}{2}\right)\right], \tag{6–35}$$

$$\begin{aligned}
\frac{\partial S(r,\psi)}{\partial r} &= -\frac{t^2}{R}\left[\frac{1 - t^2}{D^3} + \frac{4}{D} + 1 - 6D\right.\\
&\quad \left. - t\cos\psi\left(13 + 6\ln\frac{1 - t\cos\psi + D}{2}\right)\right],\\
\frac{\partial S(r,\psi)}{\partial \psi} &= -t^2\sin\psi\left[\frac{2}{D^3} + \frac{6}{D} - 8\right.\\
&\quad \left. - 3\,\frac{1 - t\cos\psi - D}{D\sin^2\psi} - 3\ln\frac{1 - t\cos\psi + D}{2}\right].
\end{aligned} \tag{6–36}$$

These expressions are used in (6–21) and (6–30) to compute T and $\delta\mathbf{g}$.

The separation N_P of the geopotential surface through P, $W = W_P$, and the corresponding spheropotential surface $U = W_P$ is according to Bruns' theorem given by

$$N_P = \frac{T_P}{\gamma_Q}\,; \tag{6–37}$$

see also Sect. 2.14 and Fig. 2.15.

The deflection of the vertical, which is the deviation of the actual plumb line from the normal plumb line at P, is represented by its north-south and east-west components,

$$\xi_P = -\frac{1}{r}\,\frac{\partial N_P}{\partial \bar{\varphi}}\,, \qquad \eta_P = -\frac{1}{r\,\cos\bar{\varphi}}\,\frac{\partial N_P}{\partial \lambda}\,; \tag{6–38}$$

these equations correspond to (2–377). Since γ varies very little with latitude and is independent of longitude, we have

$$\frac{\partial N_P}{\partial \bar{\varphi}} = \frac{\partial}{\partial \bar{\varphi}}\left(\frac{T_P}{\gamma_Q}\right) = \frac{1}{\gamma_Q}\,\frac{\partial T_P}{\partial \bar{\varphi}} - \frac{T_P}{\gamma_Q^2}\,\frac{\partial \gamma_Q}{\partial \bar{\varphi}} \doteq \frac{1}{\gamma_Q}\,\frac{\partial T_P}{\partial \bar{\varphi}} \tag{6–39}$$

and

$$\frac{\partial N_P}{\partial \lambda} = \frac{1}{\gamma_Q}\,\frac{\partial T_P}{\partial \lambda}\,. \tag{6–40}$$

Substituting the results of (6–39) and (6–40) into (6–38) and comparing then with (6–20) shows that

$$\xi_P = -\frac{1}{\gamma_Q}\,\delta g_{\bar{\varphi}}\,, \qquad \eta_P = -\frac{1}{\gamma_Q}\,\delta g_{\lambda}\,. \tag{6–41}$$

We see that N_P, ξ_P, η_P are given by Eqs. (6–21) and (6–30), apart from the factor $\pm 1/\gamma_Q$. Hence, these equations are the extensions of Stokes' and Vening Meinesz' formulas for points outside the earth and reduce to these formulas for $r = R$, $t = 1$.

Writing Eqs. (6–41) in the form

$$\delta g_{\bar{\varphi}} = -\gamma\,\xi\,, \qquad \delta g_{\lambda} = -\gamma\,\eta\,, \tag{6–42}$$

we see that the horizontal components of $\delta\mathbf{g}$ are directly related to the deflection of the vertical, which is the difference *in direction* of the vectors $\mathbf{g}$ and $\boldsymbol{\gamma}$. The radial component δg_r, however, represents the difference *in magnitude* of these vectors, since as a spherical approximation

$$-\delta g_r = \delta g = g_P - \gamma_P\,, \tag{6–43}$$

which is the scalar gravity disturbance (see Sect. 2.12).

Note that here the gravity disturbance δg is the basic quantity to be computed, rather than the gravity anomaly Δg, because both g and γ refer to the computation point P.

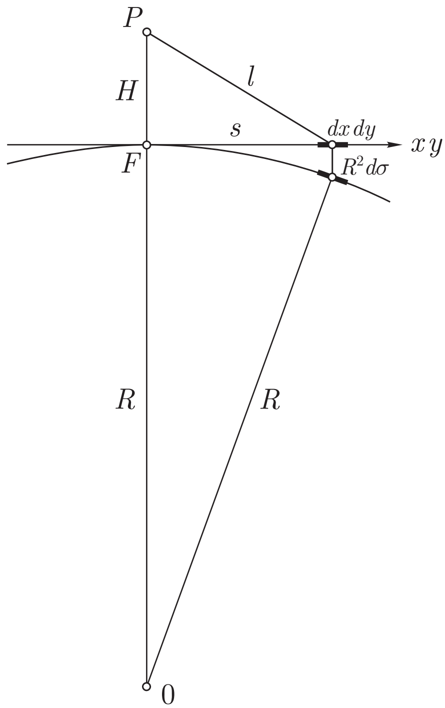

Fig. 6.2. Plane approximation

6.4 Gravity disturbances by upward continuation

We apply Poisson's integral formula (1–123) to the harmonic function T:

$$T_P = \frac{R\,(r^2 - R^2)}{4\pi} \iint_\sigma \frac{T}{l^3}\, d\sigma\,. \tag{6–44}$$

In the neighborhood of P (Fig. 6.2), the sphere practically coincides with its tangent plane at F. Since the value of the integrand is very small at greater distances from P, we may extend the integration over the tangent plane instead of over the sphere. Then, according to Fig. 6.2,

$$l = \sqrt{s^2 + H^2}\,. \tag{6–45}$$

We introduce a rectangular coordinate system x, y, z, the x-axis pointing north and the y-axis pointing east in the tangent plane. Then we may also write

$$l = \sqrt{x^2 + y^2 + H^2}\,, \tag{6–46}$$

the surface element becomes

$$R^2\, d\sigma \doteq dx\, dy\,, \tag{6–47}$$

and we further have

$$\begin{aligned} r &= R + H\,, \\ r^2 - R^2 &= (r+R)(r-R) \doteq 2R\,H\,. \end{aligned} \tag{6–48}$$

Thus, (6–44) becomes the plane formula

$$T_P = \frac{H}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{T}{l^3}\,dx\,dy = \frac{H}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{T}{(x^2+y^2+H^2)^{3/2}}\,dx\,dy\,. \tag{6–49}$$

This important formula is called the *"upward continuation integral"*. It performs the computation of the value of the harmonic function T at a point above the xy-plane from the values of T given on the plane, that is, the upward continuation of a harmonic function. Both T and its partial derivatives, $\partial T/\partial x$, $\partial T/\partial y$, $\partial T/\partial z$, are harmonic, because if

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0\,, \tag{6–50}$$

then we also have

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = 0\,. \tag{6–51}$$

Thus, the upward continuation integral (6–49), which applies for any harmonic function, may also be applied to $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial z$.

As T is the disturbing potential, its partial derivatives are the components of the gravity disturbance:

$$\frac{\partial T}{\partial x} = \delta g_{\bar\varphi}\,, \qquad \frac{\partial T}{\partial y} = \delta g_{\lambda}\,, \qquad \frac{\partial T}{\partial z} = \delta g_r\,. \tag{6–52}$$

We are not writing δg_x, δg_y, δg_z because we wish to reserve this notation for the components in the geocentric global coordinate system, which should not be confused with the local system introduced in this section. As usual, $r, \bar\varphi, \lambda$ denote geocentric spherical coordinates (see Sect. 6.3) corresponding to the spherical approximation.

Thus, we have in addition to (6–49)

$$\delta g_r = \frac{H}{2\pi}\iint\limits_{-\infty}^{\infty}\frac{\delta g_r}{l^3}\,dx\,dy\,, \tag{6–53}$$

$$\delta g_{\bar{\varphi}} = \frac{H}{2\pi} \iint\limits_{-\infty}^{\infty} \frac{\delta g_{\bar{\varphi}}}{l^3}\, dx\, dy\,,$$
$$\delta g_{\lambda} = \frac{H}{2\pi} \iint\limits_{-\infty}^{\infty} \frac{\delta g_{\lambda}}{l^3}\, dx\, dy\,. \tag{6–54}$$

On the left-hand side of these equations, the components of $\delta\mathbf{g}$ refer to the elevated point P; in the integral on the right-hand side, they are taken at sea level and are to be computed from the expressions

$$\delta g_r = -\delta g = -\left(\Delta g + \frac{2\gamma_0}{R}\, N\right)\,, \tag{6–55}$$

$$\delta g_{\bar{\varphi}} = -\gamma_0\, \xi\,,$$
$$\delta g_{\lambda} = -\gamma_0\, \eta\,, \tag{6–56}$$

which follow from (2–264) together with (6–42) and (6–43) applied to sea level. The symbols R and γ_0 denote, as usual, a mean earth radius and a mean value of gravity on the earth's surface.

Hence, we may compute T and $\delta\mathbf{g}$ by means of the upward continuation integral if the geoidal undulations N and the deflection components ξ and η at the earth's surface are given.

The plane approximation is sufficient except for very high altitudes (e.g., > 250 km). Otherwise, we must use the spherical formula (6–44) for T. For the radial component δg_r, formula (6–44) may also be applied with T replaced by $r\,\delta g$, since $r\,\delta g$ and $r\,\Delta g$ are harmonic as we know from Sect. 2.14. The corresponding spherical formulas for the upward continuation of the horizontal components $\delta g_{\bar{\varphi}}$ and δg_{λ} are not known. The reason why the same formula, the upward continuation integral, applies for T and the components of $\delta\mathbf{g}$ in the planar case only is that the derivatives of T are harmonic only when referred to a Cartesian coordinate system.

6.5 Additional considerations

Reference surface

The preceding formulas for the disturbing potential T and the gravity disturbance vector $\delta\mathbf{g}$ are rigorously valid if the reference surface is a sphere. In practice, the gravity anomalies are referred to an ellipsoid. The above formulas for T and $\delta\mathbf{g}$ are also valid for an ellipsoidal reference surface if a relative error of the order of the flattening $f \doteq 0.3\%$ is neglected, that is, as a spherical approximation. The reader is reminded that *this does not mean* *that the ellipsoid is replaced by a sphere in any geometrical sense*; rather it means that in the originally elliptical formulas the first and higher powers of the flattening are neglected, whereby they formally become spherical formulas.

Since the gravity anomalies, etc., are referred to an ellipsoid, we must be very careful in computing t, which enters into the formulas of Sect. 6.3. If an exact sphere of radius R were used as a reference surface, then we should have $r = R + H$, where H is the elevation of the computation point above the sphere. Actually, we use a reference ellipsoid; then we again have

$$r = R + H\,, \qquad t = \frac{R}{R+H}\,, \tag{6–57}$$

but H is now the elevation *above the ellipsoid* (or, to a sufficient accuracy, *above sea level*), the constant $R = 6371$ km being the earth's mean radius. Thus, r as computed by (6–57) differs from the geocentric radius vector $r = \sqrt{x^2 + y^2 + z^2}$. We have already mentioned that we may replace the geocentric latitude $\bar{\varphi}$ by the ellipsoidal latitude φ, as far as T and $\delta\mathbf{g}$ are concerned – for instance, by putting $\bar{\varphi} = \varphi$ in (6–26) or (6–29).

Data

For all computations dealing with the external gravity field of the earth, *free-air gravity anomalies* must be used for Δg, since all other types of gravity anomalies correspond to some removal or transport of masses whereby the external field is changed. If, in addition to Δg, deflections of the vertical ξ, η (in the upward continuation) are used, then these quantities should be computed from free-air anomalies. If, as usually done, the normal free-air gradient $\partial y/\partial h \doteq 0.3086$ mgal/m is used for the free-air reduction, then the free-air anomalies refer, strictly speaking, to the earth's physical surface (to ground level) rather than to the geoid (to sea level). The N values computed from them by Stokes' formula are height anomalies ζ, referring to the ground, rather than heights of the actual geoid. However, this distinction is insignificant and can be ignored in most cases, so that we may consider Δg as sea-level anomalies (see Sect. 8.6).

If we cannot neglect this distinction, aiming at highest accuracy in high and steep mountains for low altitudes H, then we may proceed as follows. We reduce the free-air anomaly Δg from the ground point A to the corresponding point A_0 at sea level (Fig. 6.3):

$$\Delta g^{\text{harmonic}} = \Delta g - \frac{\partial \Delta g}{\partial h}\, h\,, \tag{6–58}$$

and use the sea level anomaly $\Delta g^{\text{harmonic}}$ so obtained. The vertical gradient $\partial\Delta g/\partial h$ may be computed by applying formula (2–394) using the ground-

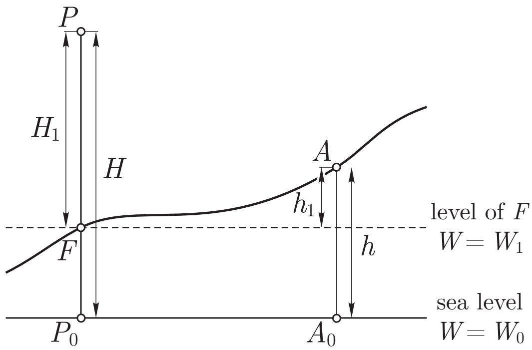

Fig. 6.3. Reduction to sea level and to the level of F

level anomalies Δg. Or we may reduce to any other level surface $W = W_1$, for instance, to that passing through F (Fig. 6.3), using h_1 instead of h in (6–58). Then we should also use H_1, rather than H, in (6–57). For large-scale purposes, reduction to sea level appears to be preferable. Probably such a reduction will attain a considerable amount only in exceptional cases so that it can usually be neglected and H in the formulas of Sects. 6.3 and 6.4 may be taken as the height of P above sea level or above ground. See also Sect. 8.6.

Computation of the gravity vector

After computing the components δg_r, $\delta g_{\bar{\varphi}}$, δg_λ by numerical integration, we may transform them into Cartesian coordinates δg_x, δg_y, δg_z with respect to the global coordinate system.

We may go via ellipsoidal-harmonic coordinates according to Sect. 6.2. For the small quantities δg_u, δg_β, δg_λ, we may apply the spherical approximation, neglecting a relative error of the order of the flattening. If the flattening is neglected, then the ellipsoidal-harmonic coordinates u, β, λ reduce to the spherical coordinates r, $\bar{\varphi}$, λ so that as a spherical approximation

$$\delta g_u = \delta g_r \, , \qquad \delta g_\beta = \delta g_{\bar{\varphi}} \, , \tag{6–59}$$

δg_λ being rigorously the same in both systems. Thus, δg_r, $\delta g_{\bar{\varphi}}$, δg_λ may also be considered as the components of $\delta \mathbf{g}$ in ellipsoidal-harmonic coordinates.

Then we have

$$g_u = \gamma_u + \delta g_r \, , \qquad g_\beta = \gamma_\beta + \delta g_{\bar{\varphi}} \, , \qquad g_\lambda = \delta g_\lambda \, ; \tag{6–60}$$

and g_x, g_y, g_z are obtained by (6–17), the components of $\mathbf{g}$ replacing the corresponding components of $\boldsymbol{\gamma}$. It is evident that the spherical approximation can only be used for $\delta\mathbf{g}$ so that γ_u and γ_β *must be computed by the rigorous formulas* (6–12).

The gravity potential W may be computed by the first equation of (6–4); the gravitational potential V is obtained by subtracting the centrifugal potential $\omega^2(x^2 + y^2)/2$; and the vector of gravitation is given by (6–5).

6.6 Gravity anomalies and disturbances compared

Suppose gravity g is to be computed at some point P outside the earth (Fig. 6.4); we consider here only the *magnitude* of the gravity vector. This is conveniently done by adding a correction to the normal gravity γ. From Sect. 2.12 and later, we recall the two different kinds of such a correction, $g - \gamma$:

1. the *gravity disturbance* δg, in which g and γ both refer to the same point P;
2. the *gravity anomaly* Δg, in which g refers to P, but γ refers to the corresponding point Q, which is situated on the plumb line of P and whose normal potential U is the same as the actual potential W of P, that is, $U_Q = W_P$.

These two quantities are connected by

$$\Delta g = \delta g - \frac{2\gamma_0}{R} N_P \,; \qquad (6\text{–}61)$$

this simple relation is sufficient for moderate altitudes.

The gravity disturbance is used when the spatial position of P is given, that is, its geocentric rectangular coordinates x, y, z are measured. With GPS measurements of the position of the aircraft, the use of gravity disturbances is natural.

The use of gravity anomalies Δg had been traditional. This is the case, for instance, in airborne gravity measurements, where the height of the aircraft

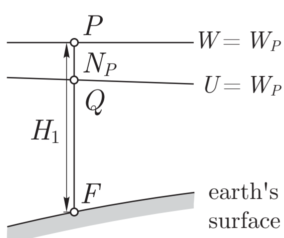

Fig. 6.4. Gravity anomalies and disturbancies

above ground is measured. This case seems rather to belong to the past. If the case should arise, gravity anomalies Δg can be upward continued just as δg as described in Sect. 6.4.

Again, free-air anomalies referred to ground level or, more accurately, to some level surface, are to be used. If the ground is elevated above sea level but reasonably flat, it is somewhat better to regard H as elevation above ground rather than above sea level, because the ground may then be considered locally part of a level surface.

The inverse problem, the downward continuation of gravity anomalies or rather gravity disturbances, occurs in the reduction of gravity measured on board an aircraft. There is, of course, a relation to harmonic downward continuation in the solution of Molodensky's problem as described in Sect. 8.6.

Upward and downward continuation are also tools of geophysical exploration, but here the objective is quite different. Several methods have been developed in this connection, some of which are also applicable for geodetic purposes; see, e.g., Dobrin and Savit (1988) or Telfort et al. (1990).

Upward and downward continuation are related as direct and inverse problems in the theory of inverse problems, see Anger et al. (1993) and also www.inas.tugraz.at under forschung/InverseProblems/AngerMoritz.html, where additional references can be found.