4 Heights

4.1 Spirit leveling

The principle of spirit leveling is well known. To measure the height difference δH_{AB} between two points A and B, vertical rods are set up at each of these two points and a level (leveling instrument) somewhere between them (Fig. 4.1). Since the line $\bar{A}\,\bar{B}$ is horizontal, the difference in the rod readings $l_1 = A\bar{A}$ and $l_2 = B\bar{B}$ is the height difference:

$$\delta H_{AB} = l_1 - l_2 \,. \tag{4–1}$$

If we measure a circuit, that is, a closed leveling line where we finally return to the initial point, then the algebraic sum of all measured differences in height will not in general be rigorously zero, as one would expect, even if we had been able to observe with perfect precision. This misclosure indicates that leveling is more complicated than it appears at first sight.

Let us look into the matter more closely. Figure 4.2 shows the relevant geometrical principles. Let the points A and B be so far apart that the procedure of Fig. 4.1 must be applied repeatedly. Then the sum of the leveled height differences between A and B will not be equal to the difference in the orthometric heights H_A and H_B. The reason is that the leveling increment δn, as we henceforth denote it, is different from the corresponding increment δH_B of H_B (Fig. 4.2), due to the nonparallelism of the level surfaces. Denoting the corresponding increment of the potential W by δW, we have by (2–21)

$$-\delta W = g\,\delta n = g'\,\delta H_B \,, \tag{4–2}$$

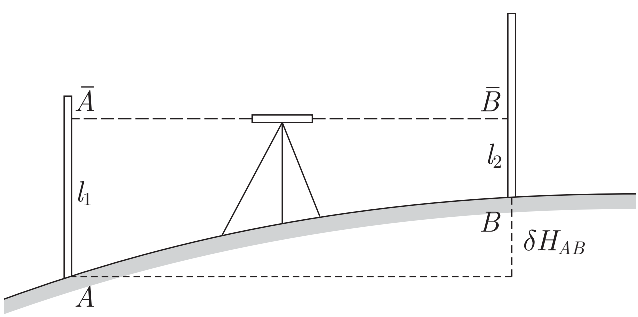

Fig. 4.1. Spirit leveling

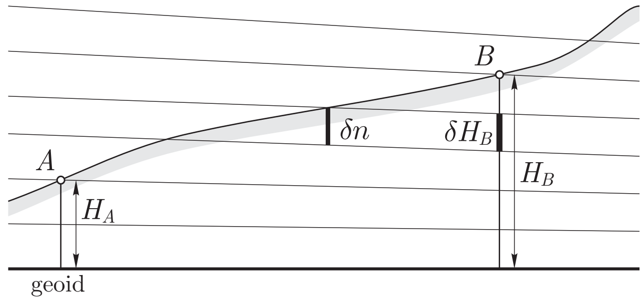

Fig. 4.2. Leveling and orthometric height

where g is the gravity at the leveling station and g' is the gravity on the plumb line of B at δH_B. Hence,

$$\delta H_B = \frac{g}{g'}\,\delta n \neq \delta n\,. \tag{4–3}$$

There is, thus, no direct geometrical relation between the result of leveling and the orthometric height, since (4–3) expresses a physical relation. What, then, if not height, is directly obtained by leveling? If gravity g is also measured, then

$$\delta W = -g\,\delta n \tag{4–4}$$

is determined, so that we obtain

$$W_B - W_A = -\sum_A^B g\,\delta n\,. \tag{4–5}$$

Thus, leveling combined with gravity measurements furnishes *potential differences*, that is, physical quantities.

It is somewhat more rigorous theoretically to replace the sum in (4–5) by an integral, obtaining

$$W_B - W_A = -\int_A^B g\,dn\,. \tag{4–6}$$

Note that this integral is independent of the path of integration; that is, different leveling lines connecting the points A and B (Fig. 4.3) should give the same result. This is evident because W is a function of position only; therefore, to every point there corresponds a unique value W. If the leveling line returns to A, then the total integral must be zero:

$$\oint g\,dn = -W_A + W_A = 0\,. \tag{4–7}$$

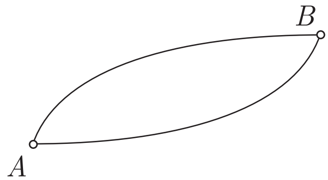

Fig. 4.3. Two different leveling lines connecting A and B (taken together, they form a circuit)

The symbol $\oint$ denotes an integral over a circuit.

On the other hand, the measured height difference, that is, the sum of the leveling increments

$$\Delta n_{AB} = \sum_{A}^{B} \delta n = \int_{A}^{B} dn \,, \tag{4–8}$$

depends on the path of integration and is, thus, not in general zero for a circuit:

$$\oint dn = \text{misclosure} \neq 0 \,. \tag{4–9}$$

In mathematical terms, dn is not a perfect differential (the differential of a function of position), whereas $dW = -g\,dn$ is perfect, so that dn becomes a perfect differential when it is multiplied by the *integrating factor* $(-g)$.

Thus, potential differences are the result of leveling combined with gravity measurements. They are basic to the whole theory of heights; even orthometric heights must be considered as quantities derived from potential differences. Leveling without gravity measurements, although applied in practice, is meaningless from a rigorous point of view, for the use of leveled heights (4–8) as such leads to contradictions (misclosures); it will not be considered here.

4.2 Geopotential numbers and dynamic heights

Let O be a point at sea level, that is, simplifying speaking, on the geoid; usually a suitable point on the seashore is taken. Let A be another point, connected to O by a leveling line. Then, by formula (4–6), the potential difference between A and O can be determined. The integral

$$\int_{0}^{A} g\,dn = W_0 - W_A = C \,, \tag{4–10}$$

which is the difference between the potential at the geoid and the potential at the point A, has been introduced as the *geopotential number* of A in Sect. 2.4. It is defined so as to be always positive.

As a potential difference, the geopotential number C is independent of the particular leveling line used for relating the point to sea level. It is the same for all points of a level surface; it can, thus, be considered as a *natural measure of height*, even if it does not have the dimension of a length.

The geopotential number C is measured in geopotential units (g.p.u.), where

$$1 \text{ g.p.u.} = 1 \text{ kgal m} = 1000 \text{ gal m}. \tag{4–11}$$

Using $g \doteq 0.98$ kgal in (4–10), we get

$$C \doteq g\,H \doteq 0.98\,H\,, \tag{4–12}$$

so that the geopotential numbers in g.p.u. are almost equal to the height above sea level in meters.

The geopotential numbers were adopted at a meeting of a Subcommission of the IAG at Florence in 1955. Formerly, the *dynamic heights* were used, defined by

$$H^{\text{dyn}} = \frac{C}{\gamma_0}\,, \tag{4–13}$$

where γ_0 is normal gravity for an arbitrary standard latitude, usually 45°:

$$\gamma_{45^\circ} = 9.806\,199\,203 \text{ m s}^{-2} = 980.6\,199\,203 \text{ gal} \tag{4–14}$$

for the GRS 1980. Just note and keep in mind that $1\,\text{gal} = 10^{-2}\ \text{m s}^{-2}$ and, accordingly, $1\,\text{mgal} = 10^{-5}\ \text{m s}^{-2}$.

The dynamic height differs from the geopotential number only in the scale or the unit: The division by the constant γ_0 in (4–13) merely converts a geopotential number into a length. However, the dynamic height has no geometrical meaning whatsoever, so that the division by an arbitrary γ_0 somehow obscures the true physical meaning of a potential difference. Hence, the geopotential numbers are, for reasons of theory and for practically establishing a national or continental height system, preferable to the dynamic heights.

Dynamic correction

It is sometimes convenient to convert the measured height difference Δn_{AB} of (4–8) into a difference of dynamic height by adding a small correction.

Using Eqs. (4–13) and (4–10) gives

$$\Delta H^{\text{dyn}}_{AB} = H^{\text{dyn}}_B - H^{\text{dyn}}_A = \frac{1}{\gamma_0}\,(C_B - C_A) = \frac{1}{\gamma_0}\int_A^B g\,dn\,, \tag{4–15}$$

which may be rewritten as

$$\Delta H_{AB}^{\text{dyn}} = \frac{1}{\gamma_0} \int_A^B (g - \gamma_0 + \gamma_0)\, dn = \int_A^B dn + \int_A^B \frac{g - \gamma_0}{\gamma_0}\, dn\,, \qquad (4\text{–}16)$$

so that

$$\Delta H_{AB}^{\text{dyn}} = \Delta n_{AB} + \mathrm{DC}_{AB}\,, \qquad (4\text{–}17)$$

where

$$\mathrm{DC}_{AB} = \int_A^B \frac{g - \gamma_0}{\gamma_0}\, dn \doteq \sum_A^B \frac{g - \gamma_0}{\gamma_0}\, \delta n \qquad (4\text{–}18)$$

is the *dynamic correction.*

As a matter of fact, the dynamic correction may also be used for computing differences of geopotential numbers. We at once obtain

$$C_B - C_A = \gamma_0\, \Delta n_{AB} + \gamma_0\, \mathrm{DC}_{AB}\,. \qquad (4\text{–}19)$$

4.3 Orthometric heights

We denote the intersection of the geoid and the plumb line through point P by P_0 (Fig. 4.4). Let C be the geopotential number of P, that is,

$$C = W_0 - W\,, \qquad (4\text{–}20)$$

and H its orthometric height, that is, the length of the plumb-line segment between P_0 and P. Perform the integration in (4–10) along the plumb line P_0P. This is permitted because the result is independent of the path. We then get

$$C = \int_0^H g\, dH\,. \qquad (4\text{–}21)$$

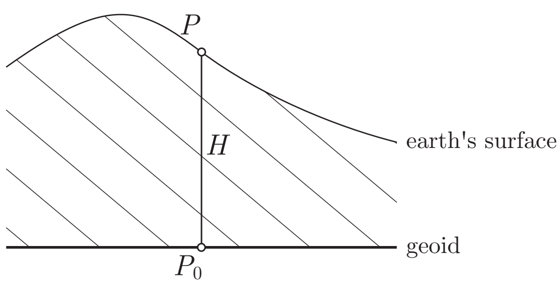

Fig. 4.4. Gravity reduction

This equation contains H in an implicit way. It is also possible to get H explicitly. From

$$dC = -dW = g\,dH\,, \qquad dH = -\frac{dW}{g} = \frac{dC}{g}\,, \tag{4–22}$$

we obtain

$$H = -\int_{W_0}^{W} \frac{dW}{g} = \int_0^C \frac{dC}{g}\,. \tag{4–23}$$

As before, the integration is extended over the plumb line.

The explicit formula (4–23), however, is of little practical use. It is better to transform (4–21) in a way that at first looks entirely trivial:

$$C = \int_0^H g\,dH = H \cdot \frac{1}{H}\int_0^H g\,dH\,, \tag{4–24}$$

so that

$$C = \bar{g}\,H\,, \tag{4–25}$$

where

$$\bar{g} = \frac{1}{H}\int_0^H g\,dH \tag{4–26}$$

is the mean value of the gravity along the plumb line between the geoid, point P_0, and the surface point P. From (4–25) it follows that

$$H = \frac{C}{\bar{g}}\,, \tag{4–27}$$

which permits H to be computed if the mean gravity $\bar{g}$ is known. Since $\bar{g}$ does not strongly depend on H, Eq. (4–27) is a practically useful formula and not merely a tautology. For determining the mean gravity $\bar{g}$, Eq. (4–26) may be written

$$\bar{g} = \frac{1}{H}\int_0^H g(z)\,dz\,, \tag{4–28}$$

where $g(z)$ is the actual gravity at the variable point Q which has the height z (Fig. 3.8). The simplest approximation is to use the simplified Prey reduction of (3–45):

$$g(z) = g + 0.0848\,(H - z)\,, \tag{4–29}$$

where g is the gravity measured at the surface point P. The integration (4–28) can now be performed immediately, giving

$$\begin{aligned} \bar{g} &= \frac{1}{H}\int_0^H \big[g + 0.0848\,(H - z)\big]\,dz \\ &= g + \frac{0.0848}{H}\left[H\,z - \frac{z^2}{2}\right]\Big|_0^H \end{aligned} \tag{4–30}$$

or

$$\bar{g} = g + 0.0424\,H \quad (g \text{ in gal, } H \text{ in km})\,. \tag{4–31}$$

The factor 0.0424 refers to the normal density $\varrho = 2.67\ \mathrm{g/cm^3}$. The corresponding formula for arbitrary constant density is, by (3–43),

$$\bar{g} = g - \left(\frac{1}{2}\frac{\partial\gamma}{\partial h} + 2\pi\,G\,\varrho\right) H\,. \tag{4–32}$$

If we use $\bar{g}$ according to (4–31) or (4–32) in the basic formula (4–27), we obtain the so-called Helmert height:

$$H = \frac{C}{g + 0.0424\,H} \tag{4–33}$$

with C in g.p.u., g in gal and H in km.

As we have seen in Sect. 3.5, this approximation replaces the terrain with an infinite Bouguer plate of constant density and of height H. This is often sufficient. Sometimes, in high mountains and for highest precision, it is necessary to apply to g a more rigorous Prey reduction, such as the three steps described in Sect. 3.5. A practical and very accurate method for this purpose has been given by Niethammer in 1932. It takes the topography into account, assuming only that the free-air gradient is normal and the density is constant down to the geoid.

It is also sufficient to calculate $\bar{g}$ as the arithmetic mean of gravity g measured at the surface point P and of gravity g_0 computed at the corresponding geoidal point P_0 by the Prey reduction:

$$\bar{g} = \tfrac{1}{2}\,(g + g_0)\,. \tag{4–34}$$

This presupposes that gravity g varies linearly along the plumb line. This can usually be assumed with sufficient accuracy, even in extreme cases, as shown by Mader (1954) and by Ledersteger (1955).

Orthometric correction

The orthometric correction is added to the measured height difference, in order to convert it into a difference in orthometric height.

We let the leveling line connect two points A and B (Fig. 4.5) and apply a simple trick first:

$$\begin{aligned}\Delta H_{AB} &= H_B - H_A = H_B - H_A - H_B^{\mathrm{dyn}} + H_A^{\mathrm{dyn}} + (H_B^{\mathrm{dyn}} - H_A^{\mathrm{dyn}})\\ &= \Delta H_{AB}^{\mathrm{dyn}} + (H_B - H_B^{\mathrm{dyn}}) - (H_A - H_A^{\mathrm{dyn}})\,.\end{aligned} \tag{4–35}$$

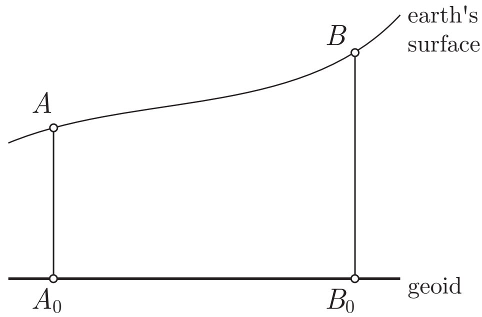

Fig. 4.5. Orthometric and dynamic correction

From (4–17), we have

$$\Delta H^{\mathrm{dyn}}_{AB} = \Delta n_{AB} + \mathrm{DC}_{AB} \,. \tag{4–36}$$

Consider now the differences between the orthometric and dynamic heights, $H_A - H^{\mathrm{dyn}}_A$ and $H_B - H^{\mathrm{dyn}}_B$. Imagine a fictitious leveling line leading from point A_0 at the geoid to the surface point A along the plumb line. Then the measured height difference would be H_A itself: $\Delta n_{A_0 A} = H_A$, so that

$$\mathrm{DC}_{A_0 A} = \Delta H^{\mathrm{dyn}}_{A_0 A} - \Delta n_{A_0 A} = H^{\mathrm{dyn}}_A - H_A \tag{4–37}$$

and

$$\begin{aligned} H_A - H^{\mathrm{dyn}}_A &= -\mathrm{DC}_{A_0 A} \,, \\ H_B - H^{\mathrm{dyn}}_B &= -\mathrm{DC}_{B_0 B} \,. \end{aligned} \tag{4–38}$$

Substituting (4–36) and (4–38) into (4–35), we finally have

$$\Delta H_{AB} = \Delta n_{AB} + \mathrm{DC}_{AB} + \mathrm{DC}_{A_0 A} - \mathrm{DC}_{B_0 B} \tag{4–39}$$

or

$$\Delta H_{AB} = \Delta n_{AB} + \mathrm{OC}_{AB} \,, \tag{4–40}$$

where

$$\mathrm{OC}_{AB} = \mathrm{DC}_{AB} + \mathrm{DC}_{A_0 A} - \mathrm{DC}_{B_0 B} \tag{4–41}$$

is the orthometric correction. This is a remarkable relation between the orthometric and dynamic corrections (Ledersteger 1955). We may write this

$$\mathrm{OC}_{AB} = \mathrm{DC}_{AB} + \mathrm{DC}_{A_0 A} + \mathrm{DC}_{B B_0} \,, \tag{4–42}$$

where we have reversed the sequence of the subscripts of the last term and, consequently, the sign. With $\mathrm{DC}_{B_0 A_0} = 0$ (why?), we may write

$$\mathrm{OC}_{AB} = \mathrm{DC}_{AB} + \mathrm{DC}_{B B_0} + \mathrm{DC}_{B_0 A_0} + \mathrm{DC}_{A_0 A} \,. \tag{4–43}$$

Accordingly, this may be written

$$\mathrm{OC}_{AB} = \mathrm{DC}_{ABB_0A_0A}. \tag{4–44}$$

Thus, the orthometric correction from A to B equals the dynamic correction over the loop ABB_0A_0A, a curious, but practically completely useless relation equivalent to (4–41). (*Question:* Why is this independent of γ_0?)

From (4–18), we find

$$\begin{aligned} \mathrm{DC}_{AB} &= \int_A^B \frac{g-\gamma_0}{\gamma_0}\, dn = \sum_A^B \frac{g-\gamma_0}{\gamma_0}\, \delta n\,, \\ \mathrm{DC}_{A_0A} &= \int_{A_0}^A \frac{g-\gamma_0}{\gamma_0}\, dH = \frac{\bar{g}_A-\gamma_0}{\gamma_0}\, H_A\,, \\ \mathrm{DC}_{B_0B} &= \int_{B_0}^B \frac{g-\gamma_0}{\gamma_0}\, dH = \frac{\bar{g}_B-\gamma_0}{\gamma_0}\, H_B\,, \end{aligned} \tag{4–45}$$

where $\bar{g}_A$ and $\bar{g}_B$ are the mean values of gravity along the plumb lines of A and B. Thus, the orthometric correction (4–41) becomes

$$\mathrm{OC}_{AB} = \sum_A^B \frac{g-\gamma_0}{\gamma_0}\, \delta n + \frac{\bar{g}_A-\gamma_0}{\gamma_0}\, H_A - \frac{\bar{g}_B-\gamma_0}{\gamma_0}\, H_B\,. \tag{4–46}$$

Here again we need the mean values of gravity along the plumb lines, $\bar{g}_A$ and $\bar{g}_B$; γ_0 is an arbitrary constant for which we always take normal gravity at 45° latitude.

Accuracy

Let us first evaluate the effect on H of an error in the mean gravity $\bar{g}$. From $H = C/\bar{g}$, we obtain by differentiation

$$\delta H = -\frac{C}{\bar{g}^2}\, \delta\bar{g} = -\frac{H}{\bar{g}}\, \delta\bar{g}\,. \tag{4–47}$$

Since $\bar{g}$ is about 1000 gal, we have, neglecting the minus sign, the simple formula

$$\delta H_{[\mathrm{mm}]} \doteq \delta\bar{g}_{[\mathrm{mgal}]}\, H_{[\mathrm{km}]}\,, \tag{4–48}$$

the subscripts denoting the units; δH is the error in H, caused by an error $\delta\bar{g}$ in $\bar{g}$.

For $H = 1$ km,

$$\delta H_{[\mathrm{mm}]} \doteq \delta\bar{g}_{[\mathrm{mgal}]}\,, \tag{4–49}$$

which shows that an error $\delta\bar{g}$ in the order of 100 mgal falsifies an elevation of 1000 m by only 10 cm.

Let us now estimate the effect of an error of the density ϱ on $\bar{g}$. Differentiating (4–32) and omitting the minus sign we find

$$\delta\bar{g} = 2\pi\, G\, H\, d\varrho\,. \tag{4–50}$$

If $\delta\varrho = 0.1\ \mathrm{g\,cm^{-3}}$ and $H = 1$ km, then

$$\delta\bar{g} = 4.2\ \mathrm{mgal}\,, \tag{4–51}$$

which causes an error of 4 mm in H. A density error of 0.6 $\mathrm{g/cm^3}$, which corresponds to the maximum variation of rock density occurring in practice, falsifies $H = 1000$ m by only 25 mm.

Mader (1954) has estimated the difference between the simple computation of mean gravity according to Helmert, Eq. (4–32), and more accurate methods that take the terrain correction into account. He found for Hochtor, in the Alps, $H = 2504$ m:

$$\begin{array}{lrl} \text{Helmert} & \bar{g} = 980.263 & \text{(Bouguer plate only)}, \\ \text{Niethammer} & 286 & \multirow{2}{*}{\Big\} \text{(also terrain correction)}\,.} \\ \bar{g} = \frac{1}{2}(g + g_0) & 285 & \end{array} \tag{4–52}$$

Mean gravity $\bar{g}$ according to (4–34) differs from Niethammer's value by only 1 mgal, which shows the linearity of g along the plumb line even in an extreme case. This corresponds to a difference in H of 3 mm. The simple Helmert height differs by about 6 cm from these more elaborately computed heights.

Therefore, the differences are very small even in this rather extreme case; we see that orthometric heights can be obtained with very high accuracy. This is of great importance for a discussion of the recent theory of Molodensky from a practical point of view. See Chap. 8, particularly Sect. 8.11.

4.4 Normal heights

Assume for the moment the gravity field of the earth to be normal, that is, $W = U$, $g = \gamma$, $T = 0$. On this assumption compute "orthometric heights"; they will be called *normal heights* and denoted by H^*. Thus, Eqs. (4–21) through (4–26) become

$$W_0 - W = C = \int_0^{H^*} \gamma\, dH^*\,, \tag{4–53}$$

$$H^* = \int_0^C \frac{dC}{\gamma}\,, \tag{4–54}$$

$$C = \bar{\gamma}\, H^*\,, \tag{4–55}$$

where

$$\bar{\gamma} = \frac{1}{H^*} \int_0^{H^*} \gamma\, dH^* \tag{4–56}$$

is the mean normal gravity along the plumb line.

As the normal potential U is a simple analytic function, these formulas can be evaluated straightforwards; but since the potential of the earth is evidently not normal, what does all this mean? Consider a point P on the physical surface of the earth. It has a certain potential W_P and also a certain normal potential U_P, but in general $W_P \neq U_P$. However, there is a certain point Q on the plumb line of P, such that $U_Q = W_P$; that is, the normal potential U at Q is equal to the actual potential W at P. The normal height H^* of P is nothing but the ellipsoidal height of Q above the ellipsoid, just as the orthometric height of P is the height of P above the geoid.

For more details the reader is referred to Sect. 8.3; Fig. 8.2 illustrates the geometric relations.

We now give some practical formulas for the computation of normal heights from geopotential numbers. Writing (4–56) in the form

$$\bar{\gamma} = \frac{1}{H^*} \int_0^{H^*} \gamma(z)\, dz \tag{4–57}$$

corresponding to (4–28), then we can express $\gamma(z)$ by (2–215) as

$$\gamma(z) = \gamma \left[1 - \frac{2}{a}\left(1 + f + m - 2f \sin^2\varphi\right) z + \frac{3}{a^2}\, z^2 \right], \tag{4–58}$$

where γ is the gravity at the ellipsoid, depending on the latitude φ but not on z. Thus, straightforward integration with respect to z yields

$$\begin{aligned} \bar{\gamma} &= \frac{1}{H^*}\, \gamma \left[z - \frac{2}{a}\left(1 + f + m - 2f \sin^2\varphi\right) \frac{z^2}{2} + \frac{3}{a^2}\, \frac{z^3}{3} \right] \Bigg|_0^{H^*} \\ &= \frac{1}{H^*}\, \gamma \left[H^* - \frac{1}{a}\left(1 + f + m - 2f \sin^2\varphi\right) H^{*2} + \frac{1}{a^2}\, H^{*3} \right] \end{aligned} \tag{4–59}$$

or

$$\bar{\gamma} = \gamma \left[1 - \left(1 + f + m - 2f \sin^2\varphi\right) \frac{H^{*2}}{a} + \frac{H^{*2}}{a^2} \right]. \tag{4–60}$$

This formula may be used for computing H^* by the formula

$$H^* = \frac{C}{\bar{\gamma}}\,. \tag{4–61}$$

The mean theoretical gravity itself depends on H^*, by (4–60), but not strongly, so that an iterative solution is very simple.

It is also possible to give a direct expression of H^* in terms of the geopotential number C by substituting (4–60) into (4–61) and expanding into a series of powers of H^*:

$$H^* = \frac{C}{\gamma}\left[1 + \frac{1}{a}\left(1 + f + m - 2f\sin^2\varphi\right)H^* + O(H^{*2})\right]. \tag{4–62}$$

Solving this equation for H^* and expanding H^* in powers of C/γ, we obtain

$$H^* = \frac{C}{\gamma}\left[1 + \left(1 + f + m - 2f\sin^2\varphi\right)\frac{C}{a\,\gamma} + \left(\frac{C}{a\,\gamma}\right)^2\right], \tag{4–63}$$

where γ is normal gravity at the ellipsoid, for the same latitude φ. The accuracy of this formula will be sufficient for almost all practical purposes; still more accurate expressions are given in Hirvonen (1960).

Corresponding to the dynamic and orthometric corrections, there is a *normal correction* NC of the measured height differences. Equation (4–46) immediately yields, on replacing $\bar{g}$ by $\bar{\gamma}$ and H by H^*:

$$\mathrm{NC}_{AB} = \sum_A^B \frac{g - \gamma_0}{\gamma_0}\,\delta n + \frac{\bar{\gamma}_A - \gamma_0}{\gamma_0}\,H_A^* - \frac{\bar{\gamma}_B - \gamma_0}{\gamma_0}\,H_B^*, \tag{4–64}$$

so that

$$\Delta H_{AB}^* = H_B^* - H_A^* = \Delta n_{AB} + \mathrm{NC}_{AB}. \tag{4–65}$$

The normal heights were introduced by Molodensky in connection with his method of determining the physical surface of the earth; see Chap. 8.

4.5 Comparison of different height systems

By means of the geopotential number

$$C = W_0 - W = \int_{\text{geoid}}^{\text{point}} g\,dn, \tag{4–66}$$

we can write the different kinds of height in a common form which is very instructive:

$$\text{height} = \frac{C}{G_0}, \tag{4–67}$$

where the height systems differ according to how the gravity value G_0 in the denominator is chosen. We have:

$$\begin{aligned} &\text{dynamic height:} && G_0 = \gamma_0 = \text{ constant}\,, \\ &\text{orthometric height:} && G_0 = \bar{g}\,, \\ &\text{normal height:} && G_0 = \bar{\gamma}\,. \end{aligned} \tag{4–68}$$

It is seen that one can devise an unlimited number of other height systems by selecting G_0 in a different way.

The geopotential number C is, in a way, the most direct result of leveling and is of great scientific importance. However, it is not a height in a geometrical or practical sense. While the dynamic height has at least the dimension of a height, it has no geometrical meaning. One advantage is that points of the same level surface have the same dynamic height; this corresponds to the intuitive feeling that if we move horizontally, we remain at the same height. Note that the orthometric height differs for points of the same level surface because the level surfaces are not parallel. This gives rise to the well-known paradoxes of "water flowing uphill", etc.

The dynamic correction can be very large, because gravity varies from equator to pole by about 5000 mgal. Take, for instance, a leveling line of 1000 m difference of height at the equator, where $g \doteq 978.0$ gal, computed with $\gamma_0 = \gamma_{45°} \doteq 980.6$ gal. Then (4–18) gives a dynamic correction of approximately

$$\mathrm{DC} = \frac{978.0 - 980.6}{980.6} \cdot 1000\ \mathrm{m} = -2.7\ \mathrm{m}\,. \tag{4–69}$$

Because of these large corrections, dynamic heights are not suitable as practical heights, and the geopotential numbers are preferable for scientific purposes.

Orthometric heights are the natural "heights above sea level", that is, heights above the geoid. Therefore, they have an unequalled geometrical and physical significance. Their computation is relatively laborious, unless Helmert's simple formula (4–33) is used, which is sufficient in most cases. The orthometric correction is rather small. In the Alpine leveling line of Mader (1954), leading from an elevation of 754 m to 2505 m, the orthometric correction is about 15 cm per 1 km of measured height difference. See also Sect. 8.15.

The physical and geometrical meaning of the *normal heights* is less obvious; they depend on the reference ellipsoid used. Although they are basic in the new theories of physical geodesy, they have a somewhat artificial character as compared to the orthometric heights. They are, however, easy to compute rigorously; the order of magnitude of the normal corrections is about the same as that of the orthometric corrections. In some countries they have replaced the orthometric heights in practice.

For estimates of the difference between orthometric height H and normal height H^*, we refer the reader to Sect. 8.13.

All these height systems based on C are functions of position only. There are, thus, no misclosures, as there are with measured heights. From a purely practical point of view, the desired requirements of a height system are that

1. misclosures be eliminated,
2. corrections to the measured heights be as small as possible.

Empirical height systems have been devised to give smaller corrections than either the orthometric or the normal heights. They have no clear physical significance, however, and are beyond the scope of this book.

Accuracy

Leveling is one of the most accurate geodetic measurements. A standard error of ± 0.1 mm per km distance is possible; it increases with the square root of the distance.

If the error of measurement and interpolation, etc., of gravity is negligible, then the differences in the geopotential number C can be determined with an accuracy of ± 0.1 gal m per km distance; this corresponds to ± 0.1 mm in measured height. Referring to gravity measurements, it is sufficient to measure at distances of some kilometers.

Dynamic heights and normal heights are clearly as accurate as the geopotential numbers, because normal gravity γ is errorless. Orthometric heights, however, are also affected by imperfect knowledge of density, etc., but only slightly; see the end of Sect. 4.3.

Triangulated heights

Historically and for the sake of completeness, the determination of heights by triangulation, that is, by means of zenith angles, should be mentioned.

The main problem is the atmospheric refraction affecting the zenith angles. Thus, the accuracy of triangulated heights is much less than that of leveling. Consequently, triangulated heights are not considered any longer here.

For small distances (e.g., < 1 km), trigonometric height measurements, referred to the local plumb line, have the character of a leveled height difference δn This fact may be used (with care!) to fill small gaps in a leveling network.

Remark on misclosures

All misclosures in any acceptable system of heights denoted for the moment by h (not to be confused with ellipsoidal heights) must be zero:

$$\oint dh = 0 \tag{4–70}$$

for any closed path. Height networks consisting of triangles, if computed by least-squares adjustment, thus must satisfy the condition that the sum of height differences must be zero for each triangle. Mathematically, this can be shown to be equivalent to the commutativity of second derivatives:

$$\frac{\partial^2 h}{\partial x\,\partial y} = \frac{\partial^2 h}{\partial y\,\partial x} \, . \tag{4–71}$$

4.6 GPS leveling

Spirit leveling (Fig. 4.6) is a very time-consuming operation. GPS has introduced a revolution also here. The basic equation is

$$H = h - N \, . \tag{4–72}$$

This equation relates the orthometric height H (above the geoid), the ellipsoidal height h (above the ellipsoid), and the geoidal undulation N. If any two of these quantities are measured, then the third quantity can be computed.

If h is measured by GPS, and if there exists a reliable digital geoid map of N, then the orthometric height H can be obtained immediately.

Equation (4–72) can also be used for geoid determination: if h is measured by GPS, and H is available from leveling, then the geoid N can be determined as $N = h - H$. The same principle can be applied even on the oceans as *satellite altimetry*, as we will see later in Chap. 7, e.g., Eq. (7–47).

GPS leveling implies replacing to some extent the classical leveling by GPS. Referring to Fig. 4.6 and applying (4–72) to A and B leads to

$$\begin{aligned} H_A &= h_A - N_A \, , \\ H_B &= h_B - N_B \, , \end{aligned} \tag{4–73}$$

and the height difference

$$H_B - H_A = h_B - h_A - N_B + N_A \, . \tag{4–74}$$

Introducing the notations $\delta H_{AB} = H_B - H_A$, $\delta h_{AB} = h_B - h_A$, and $\delta N_{AB} = N_B - N_A$, the relation reduces to

$$\delta H_{AB} = \delta h_{AB} - \delta N_{AB} \, . \tag{4–75}$$

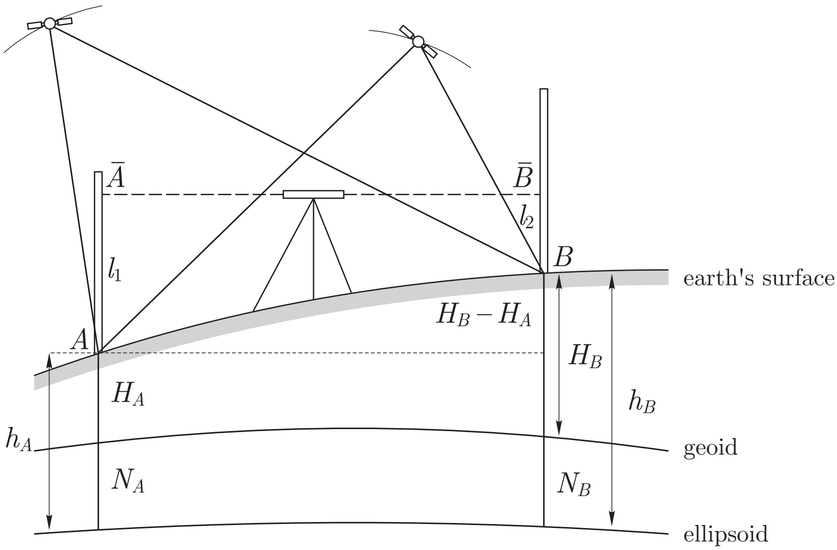

Fig. 4.6. GPS leveling

With GPS leveling, δh_{AB} is obtained, so that with a known geoid, i.e., known δN_{AB}, the orthometric height difference δH_{AB} may be computed according to (4–75). This is a tremendous advantage since otherwise the classical leveling together with gravity measurements is required to determine the orthometric height difference, see Eqs. (4–40) and (4–46).

Note that only the difference of the geoidal undulations impacts the result.