3 Gravity reduction

3.1 Introduction

Gravity g measured on the physical surface of the earth must be distinguished from normal gravity γ referring to the surface of the ellipsoid. To refer g to sea level, a reduction is necessary. Since there are masses above sea level, the reduction methods differ depending on the way how to deal with these topographic masses. Gravity reduction is essentially the same for gravity anomalies Δg and gravity disturbances δg.

Gravity reduction serves as a tool for three main purposes:

- determination of the geoid,
- interpolation and extrapolation of gravity,
- investigation of the earth's crust.

Only the first two purposes are of a direct geodetic nature. The third is of interest to theoretical geophysicists and geologists, who study the general structure of the crust, and to exploration geophysicists.

The use of Stokes' formula for the determination of the geoid requires that the gravity anomalies Δg represent boundary values at the geoid. This implies two conditions: first, gravity g must refer to the geoid; second, there must be no masses outside the geoid (Sect. 2.12). Hence, figuratively speaking, gravity reduction consists of the following steps:

1. the topographic masses outside the geoid are completely removed or shifted below sea level;
2. then the gravity station is lowered from the earth's surface (point P) to the geoid (point P_0, see Fig. 3.1).

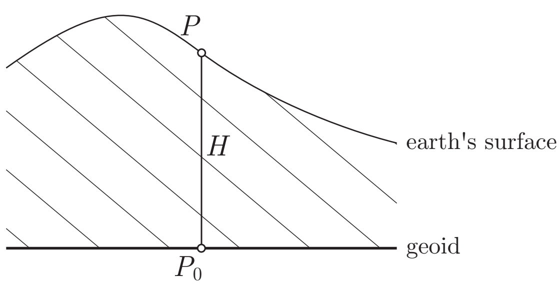

Fig. 3.1. Gravity reduction

The first step requires knowledge of the density of the topographic masses, which is somewhat problematic.

By such a reduction procedure certain irregularities in gravity due to differences in height of the stations are removed so that interpolation and even extrapolation to unobserved areas become easier (Sect. 9.7).

3.2 Auxiliary formulas

Let us compute the potential U and the vertical attraction A of a homogeneous circular cylinder of radius a and height b at a point P situated on its axis at a height c above its base (Fig. 3.2).

P outside cylinder

Assume first that P is above the cylinder, $c > b$. Then the potential is given by the general formula (1–12),

$$U = G \iiint \frac{\varrho}{l} \, dv \,. \tag{3–1}$$

Introducing polar coordinates s, α in the xy-plane by

$$x = s \cos\alpha \,, \qquad y = s \sin\alpha \,, \tag{3–2}$$

we have

$$l = \sqrt{s^2 + (c - z)^2} \tag{3–3}$$

and

$$dv = dx\, dy\, dz = s\, ds\, d\alpha\, dz \,. \tag{3–4}$$

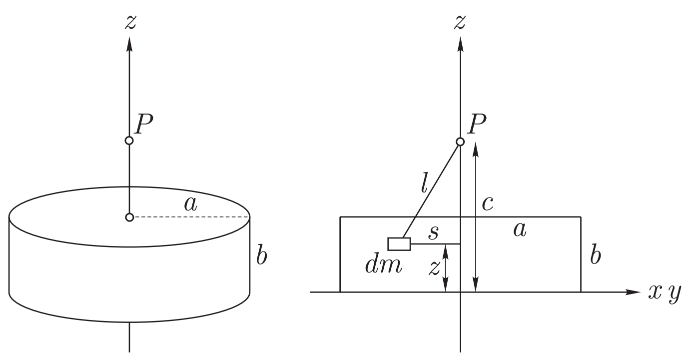

Fig. 3.2. Potential and attraction of a circular cylinder on an external point

Hence, we find, with the density $\varrho =$ constant,

$$\begin{aligned} U &= G\,\varrho \int_{\alpha=0}^{2\pi} \int_{s=0}^{a} \int_{z=0}^{b} \frac{s\;ds\;dz\;d\alpha}{\sqrt{s^2+(c-z)^2}} \\ &= 2\pi\, G\,\varrho \int_{s=0}^{a} \int_{z=0}^{b} \frac{s\;ds\;dz}{\sqrt{s^2+(c-z)^2}}\,. \end{aligned} \tag{3–5}$$

The integration with respect to s yields

$$\begin{aligned} \int_{s=0}^{a} \frac{s\;ds}{\sqrt{s^2+(c-z)^2}} &= \sqrt{s^2+(c-z)^2}\,\Big|_0^a \\ &= \sqrt{a^2+(c-z)^2} - c + z\,, \end{aligned} \tag{3–6}$$

so that we have

$$U = 2\pi\, G\,\varrho \int_0^b \Big[-c + z + \sqrt{a^2+(c-z)^2}\,\Big]\, dz\,. \tag{3–7}$$

The indefinite integral is $2\pi\, G\,\varrho$ times

$$\tfrac{1}{2}\,(c-z)^2 - \tfrac{1}{2}\,(c-z)\,\sqrt{a^2+(c-z)^2} - \tfrac{1}{2}\,a^2 \ln\Big[c-z+\sqrt{a^2+(c-z)^2}\,\Big]\,, \tag{3–8}$$

as may be verified by differentiation. Hence, U finally becomes

$$\begin{aligned} U_{\mathrm{e}} = \pi\, G\,\varrho \Big\{ &(c-b)^2 - c^2 - (c-b)\,\sqrt{a^2+(c-b)^2} + c\,\sqrt{a^2+c^2} \\ &- a^2 \ln\big[c-b+\sqrt{a^2+(c-b)^2}\,\big] + a^2 \ln\big[c+\sqrt{a^2+c^2}\,\big]\Big\}\,, \end{aligned} \tag{3–9}$$

where the subscript e denotes that P is external to the cylinder.

The vertical attraction A is the negative derivative of U with respect to the height c [see Eq. (2–22)]:

$$A = -\frac{\partial U}{\partial c}\,. \tag{3–10}$$

Differentiating (3–9), we obtain

$$A_{\mathrm{e}} = 2\pi\, G\,\varrho \Big[b + \sqrt{a^2+(c-b)^2} - \sqrt{a^2+c^2}\,\Big]\,. \tag{3–11}$$

P on cylinder

In this case we have $c = b$, and Eqs. (3–9) and (3–11) become

$$U_0 = \pi G \varrho \left[-b^2 + b\sqrt{a^2+b^2} + a^2 \ln \frac{b+\sqrt{a^2+b^2}}{a} \right], \tag{3–12}$$

$$A_0 = 2\pi G \varrho \left[a + b - \sqrt{a^2+b^2} \right]. \tag{3–13}$$

P inside cylinder

We assume that P is now inside the cylinder, $c < b$. By the plane $z = c$ we separate the cylinder into two parts, 1 and 2 (Fig. 3.3), and compute U as the sum of the contributions of these two parts:

$$U_\mathrm{i} = U_1 + U_2 , \tag{3–14}$$

where the subscript i denotes that P is now inside the cylinder. The term U_1 is given by (3–12) with b replaced by c, and U_2 by the same formula with b replaced by $b - c$. Their sum is

$$\begin{aligned} U_\mathrm{i} = \pi G \varrho \Bigg[& -c^2 - (b-c)^2 + c\sqrt{a^2+c^2} + (b-c)\sqrt{a^2+(b-c)^2} \\ & + a^2 \ln \frac{c+\sqrt{a^2+c^2}}{a} + a^2 \ln \frac{b-c+\sqrt{a^2+(b-c)^2}}{a} \Bigg]. \end{aligned} \tag{3–15}$$

It is easily seen that the attraction is the difference $A_1 - A_2$:

$$A_\mathrm{i} = 2\pi G \varrho \left[2c - b - \sqrt{a^2+c^2} + \sqrt{a^2+(b-c)^2} \right]; \tag{3–16}$$

this formula may also be obtained by differentiating (3–15) according to (3–10).

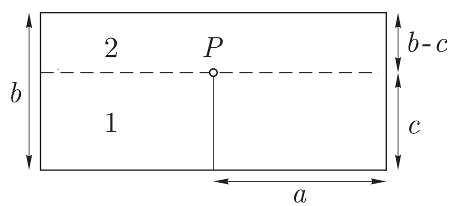

Fig. 3.3. Potential and attraction on an internal point

Circular disk

Let the thickness b of the cylinder go to zero such that the product

$$\kappa = b\,\varrho \tag{3–17}$$

remains finite. The quantity κ may then be considered as the surface density with which matter is concentrated on the surface of a circle of radius a. We need potential and attraction for an exterior point. By setting

$$\varrho = \frac{\kappa}{b} \tag{3–18}$$

in (3–9) and (3–11) and then letting $b \to 0$, we get by well-known methods of the calculus

$$\begin{aligned} U_{\mathrm{e}}^{0} &= 2\pi\, G\, \kappa \left[\sqrt{a^2+c^2} - c\right], \\ A_{\mathrm{e}}^{0} &= 2\pi\, G\, \kappa \left(1 - \frac{c}{\sqrt{a^2+c^2}}\right). \end{aligned} \tag{3–19}$$

Sectors and compartments

For a sector of radius a and angle

$$\alpha = \frac{2\pi}{n}, \tag{3–20}$$

we must divide the above formulas by n. For a compartment subtending the same angle and bounded by the radii a_1 and a_2 (Fig. 3.4), we get, in an obvious notation,

$$\begin{aligned} \Delta U &= \frac{1}{n}\left[U(a_2) - U(a_1)\right], \\ \Delta A &= \frac{1}{n}\left[A(a_2) - A(a_1)\right]. \end{aligned} \tag{3–21}$$

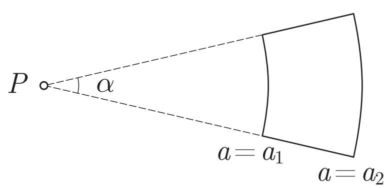

Fig. 3.4. Template compartment

Since A_e and A_i differ only by a constant, this constant drops out in the second equation of (3–21), and we obtain from (3–11) and (3–16)

$$\Delta A_\mathrm{e} = \Delta A_\mathrm{i} = \frac{2\pi}{n} G \varrho \Big[\sqrt{a_2^2 + (c-b)^2} - \sqrt{a_1^2 + (c-b)^2} - \sqrt{a_2^2 + c^2} + \sqrt{a_1^2 + c^2} \Big] . \tag{3–22}$$

On the other hand, $\Delta U_\mathrm{e} \neq \Delta U_\mathrm{i}$.

Note that we have for didactic reasons purposely used the compartments corresponding to polar coordinates (Fig. 3.4) because they are so simple and instructive, but also still useful for many purposes. For practical computation, rectangular blocks (see Fig. 2.21) are almost exclusively used. For conceptual purposes, however, the polar coordinate template remains invaluable; cf. Sect. 2.21.

3.3 Free-air reduction

For a theoretically correct reduction of gravity to the geoid, we need $\partial g/\partial H$, the vertical gradient of gravity. If g is the observed value at the surface of the earth, then the value g_0 at the geoid may be obtained as a Taylor expansion:

$$g_0 = g - \frac{\partial g}{\partial H} H \ \cdots , \tag{3–23}$$

where H is the height between P, the gravity station above the geoid, and P_0, the corresponding point on the geoid (Fig. 3.1). Suppose there are no masses above the geoid and neglecting all terms but the linear one, we have

$$g_0 = g + F , \tag{3–24}$$

where

$$F = -\frac{\partial g}{\partial H} H \tag{3–25}$$

is the *free-air reduction* to the geoid. Note that the assumption of no masses above the geoid may be interpreted in the sense that such masses have been mathematically removed beforehand, so that this reduction is indeed carried out “in free air”.

For many practical purposes it is sufficient to use instead of $\partial g/\partial H$ the normal gradient of gravity (associated with the ellipsoidal height h) $\partial\gamma/\partial h$, obtaining

$$F \doteq -\frac{\partial \gamma}{\partial h} H \doteq +0.3086\, H \ \text{[mgal]} \tag{3–26}$$

for H in meters.

3.4 Bouguer reduction

The objective of the Bouguer reduction of gravity is the complete removal of the topographic masses, that is, the masses outside the geoid.

The Bouguer plate

Assume the area around the gravity station P to be completely flat and horizontal (Fig. 3.5), and let the masses between the geoid and the earth's surface have a constant density ϱ. Then the attraction A of this so-called Bouguer plate is obtained by letting $a \to \infty$ in (3–13), since the plate, considered plane, may be regarded as a circular cylinder of thickness $b = H$ and infinite radius. By well-known rules of the calculus, we obtain

$$A_B = 2\pi G \varrho H \tag{3–27}$$

as the attraction of an infinite Bouguer plate. With standard density $\varrho = 2.67\ \mathrm{g\,cm^{-3}}$ this becomes

$$A_B = 0.1119\, H \ \text{[mgal]} \tag{3–28}$$

for H in meters.

Removing the plate is equivalent to subtracting its attraction (3–27) from the observed gravity. This is called *incomplete Bouguer reduction.* Note that this is the usual "plane" Bouguer plate; for a truly "spherical" Bouguer plate we would have 4π instead of 2π (Moritz 1990: p. 235).

To continue and complete our gravity reduction, we must now apply the free-air reduction F as given in (3–26). This combined process of removing the topographic masses and applying the free-air reduction is called *complete Bouguer reduction.* Its result is Bouguer gravity at the geoid:

$$g_B = g - A_B + F \,. \tag{3–29}$$

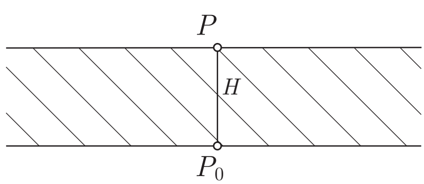

Fig. 3.5. Bouguer plate

With the assumed numerical values, we have

$$\begin{array}{ll} \text{gravity measured at } P & g \\ \text{minus Bouguer plate} & \quad -\,0.1119\ H \\ \text{plus free-air reduction} & \quad +\,0.3086\ H \\ \hline \text{Bouguer gravity at } P_0 & g_B = g + 0.1967\ H\,. \end{array} \tag{3–30}$$

Since g_B now refers to the geoid, we obtain genuine gravity anomalies in the sense of Sect. 2.12 by subtracting normal gravity γ referred to the ellipsoid:

$$\Delta g_B = g_B - \gamma\,. \tag{3–31}$$

They are called *Bouguer anomalies.*

Terrain correction

This simple procedure is refined by taking into account the deviation of the actual topography from the Bouguer plate of P (Fig. 3.6). This is called *terrain correction* or *topographic correction.* At A the mass surplus Δm_+, which attracts upward, is removed, causing g at P to increase. At B the mass deficiency Δm_- is made up, causing g at P to increase again. *The terrain correction is always positive.*

The practical determination of the terrain correction A_t is carried out by means of a template, similar to that shown in Fig. 2.20, using (3–22) and adding the effects of the individual compartments:

$$A_t = \sum \Delta A\,. \tag{3–32}$$

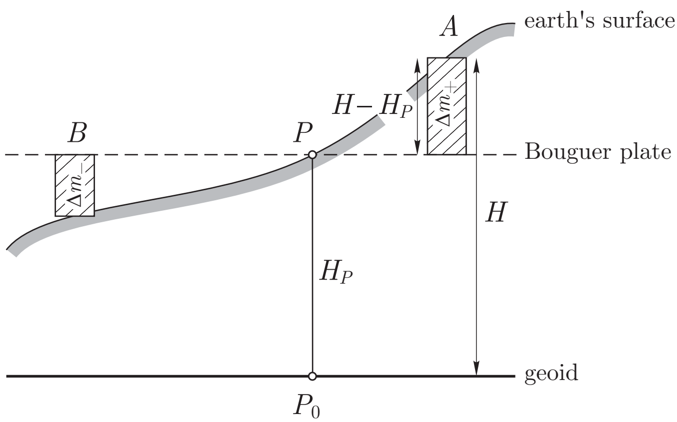

Fig. 3.6. Terrain correction

Again, we can use a template in polar coordinates (Fig. 2.20) for theoretical considerations or a rectangular grid (Fig. 2.21) for numerical computations. For a surplus mass Δm_+, $H > H_p$, we have

$$b = H - H_P\,, \qquad c = 0\,; \tag{3–33}$$

and for a mass deficiency Δm_-, $H < H_P$,

$$b = c = H_P - H\,. \tag{3–34}$$

By adding the terrain correction A_t to (3–29), the *refined Bouguer gravity*

$$g_B = g - A_B + A_t + F \tag{3–35}$$

is obtained. The Bouguer reduction and the corresponding Bouguer anomalies Δg_B are called *refined* or *simple*, depending on whether the terrain correction has been applied or not.

In practice it is convenient to separate the Bouguer reduction into the effect of a Bouguer plate and the terrain correction, because the amount of the latter is usually much less. Even for mountains 3000 m in height, the terrain correction is only of the order of 50 mgal (Heiskanen and Vening Meinesz 1958: p. 154).

Unified procedure

It is also possible to compute the total effect of the topographic masses,

$$A_T = A_B - A_t\,, \tag{3–36}$$

in one step by using columns with base at sea level (Fig. 3.7), again sub-

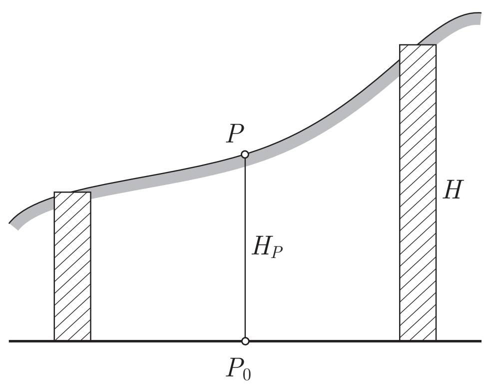

Fig. 3.7. Bouguer reduction

dividing the terrain by means of a template. Note the difference between A_T, the attraction of the topographic masses, and the terrain correction A_t! Then

$$A_T = \sum \Delta A \,, \tag{3–37}$$

where we now have $b = H$, $c = H_P$. Use (3–13) with $b = H_P$ for the innermost circle.

Instead of (3–35), we now have

$$g_B = g - A_T + F \,. \tag{3–38}$$

The Bouguer reduction may be still further refined by the consideration of density anomalies, anomalies in the free-air gradient of gravity (Sect. 2.20), and spherical effects. More computational formulas may be found in Jung (1961: Sect. 6.4).

3.5 Poincaré and Prey reduction

Suppose we need the gravity g' inside the earth. Since g' cannot be measured, it must be computed from the surface gravity. This is done by reducing the measured values of gravity according to the method of Poincaré and Prey.

We denote the point at which g' is to be computed by Q, so that $g' = g_Q$. Let P be the corresponding surface point so that P and Q are situated on the same plumb line (Fig. 3.8). Gravity at P, denoted by g_P, is measured.

The direct way of computing g_Q would be to use the formula

$$g_Q = g_P - \int_Q^P \frac{\partial g}{\partial H}\, dH \,, \tag{3–39}$$

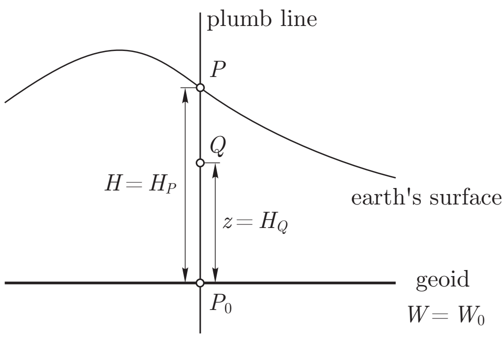

Fig. 3.8. Prey reduction

provided that the actual gravity gradient $\partial g/\partial H$ inside the earth were known. It can be obtained by Bruns' formula (2–40),

$$\frac{\partial g}{\partial H} = -2g\,J + 4\pi\,G\,\varrho - 2\omega^2\,, \tag{3–40}$$

if the mean curvature J of the geopotential surfaces and the density ϱ are known between P and Q.

The normal free-air gradient is given by (2–147):

$$\frac{\partial \gamma}{\partial h} = -2\gamma\,J_0 - 2\omega^2\,, \tag{3–41}$$

where J_0 is the mean curvature of the spheropotential surfaces. If the approximation

$$g\,J \doteq \gamma\,J_0 \tag{3–42}$$

is sufficient, then we get from (3–40) and (3–41)

$$\frac{\partial g}{\partial H} = \frac{\partial \gamma}{\partial h} + 4\pi\,G\,\varrho\,. \tag{3–43}$$

Numerically, neglecting the variation of $\partial\gamma/\partial h$ with latitude, we find for the density $\varrho = 2.67\ \mathrm{g\,cm^{-3}}$ and (truncated) $G = 6.67 \cdot 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$

$$\frac{\partial g}{\partial H} = -0.3086 + 0.2238 = -0.0848\ \mathrm{gal\,km^{-1}}\,, \tag{3–44}$$

so that (3–39) becomes

$$g_Q = g_P + 0.0848\,(H_P - H_Q) \tag{3–45}$$

with g in gal and H in km. This simple formula, although being rather crude, is often applied in practice.

The accurate way to compute g_Q would be to use (3–39) and (3–40) with the actual mean curvature J of the geopotential surfaces, but this would require knowledge of the detailed shape of these surfaces far beyond what is attainable today.

Another way of computing g_Q, which is more practicable at present, is the following. It is similar to the usual reduction of gravity to sea level (see Sect. 3.4) and consists of three steps:

1. Remove all masses above the geopotential surface $W = W_Q$, which contains Q, and subtract their attraction from g at P.

2. Since the gravity station P is now "in free air", apply the free-air reduction, thus moving the gravity station from P to Q.

3. Restore the removed masses to their former position, and add algebraically their attraction to g at Q.

The purpose of this slightly complicated but logically clear procedure is that in step 2 the *free-air* gradient can be used. If we *here* replace the actual free-air gradient by the normal gradient $\partial\gamma/\partial h$, then the error will presumably be smaller than in using (3–43).

Note that the free-air gradient can also be accurately computed alternatively by (2–394); the gravity anomalies Δg in this formula are the gravity anomalies obtained after performing step 2, that is, Bouguer anomalies referred to the lower point Q.

The effect of the masses above Q (steps 1 and 3) may be computed, e.g., by means of some kind of template or computer procedure for numerical three-dimensional integration. If the terrain correction is neglected and only the infinite Bouguer plate between P and Q of the normal density $\varrho = 2.67\ \mathrm{g\,cm^{-3}}$ is taken into account, then we obtain with the steps numbered as above:

gravity measured at P	g_P
1. remove Bouguer plate	$-\,0.1119\,(H_P - H_Q)$
2. free-air reduction from P to Q	$+\,0.3086\,(H_P - H_Q)$
3. restore Bouguer plate	$-\,0.1119\,(H_P - H_Q)$
together: gravity at Q	$g_Q = g_P + 0.0848\,(H_P - H_Q)\,.$

(3–46)

This is the same as (3–45), which is, thus, confirmed independently. We see now that the use of (3–43) or (3–45) amounts to replacing the terrain with a Bouguer plate.

Finally, we note that the reduction of Poincaré and Prey, abbreviated as *Prey reduction*, yields the actual gravity which would be measured inside the earth if this were possible. Its purpose is, thus, completely different from the purpose of the other gravity reductions which give boundary values at the geoid.

It cannot be directly used for the determination of the geoid but is needed to obtain orthometric heights as will be discussed in Sect. 4.3. Actual gravity g_0 at a geoidal point P_0 is related to Bouguer gravity g_B, Eq. (3–38), by

$$g_0 = g_B - A_{T,P_0}\,. \tag{3–47}$$

It is obtained by subtracting from g_B the attraction A_{T,P_0} of the topographic masses on P_0, which corresponds to restoring the topography after the free-air reduction of Bouguer gravity from P to P_0.

3.6 Isostatic reduction

3.6.1 Isostasy

One might be inclined to assume that the topographic masses are simply superposed on an essentially homogeneous crust. If this were the case, the Bouguer reduction would remove the main irregularities of the gravity field so that the Bouguer anomalies would be very small and would fluctuate randomly around zero. However, just the opposite is true. Bouguer anomalies in mountainous areas are systematically negative and may attain large values, increasing in magnitude on the average by 100 mgal per 1000 m of elevation. The only explanation possible is that there is some kind of mass deficiency under the mountains. This means that the topographic masses are *compensated* in some way.

There is a similar effect for the deflections of the vertical. The actual deflections are smaller than the visible topographic masses would suggest. In the middle of the nineteenth century, J.H. Pratt observed such an effect in the Himalayas. At one station in this area he computed a value of $28''$ for the deflection of the vertical from the attraction of the visible masses of the mountains. The value obtained through astrogeodetic measurements was only $5''$. Again, some kind of compensation is needed to account for this discrepancy.

Two different theories for such a compensation were developed at almost exactly the same time, by J.H. Pratt in 1854 and 1859 and by G.B. Airy in 1855. According to Pratt, the mountains have risen from the underground somewhat like a fermenting dough. According to Airy, the mountains are floating on a fluid lava of higher density (somewhat like an iceberg floating on water), so that the higher the mountain, the deeper it sinks.

Pratt–Hayford system

This system of compensation was outlined by Pratt and put into a mathematical form by J.F. Hayford, who used it systematically for geodetic purposes.

The principle is illustrated in Fig. 3.9. Underneath the level of compensation there is uniform density. Above, the mass of each column of the same cross section is equal. Let D be the depth of the level of compensation, reckoned from sea level, and let ϱ_0 be the density of a column of height D. Then the density ϱ of a column of height $D + H$ (H representing the height of the topography) satisfies the equation

$$(D + H)\,\varrho = D\,\varrho_0\,, \tag{3–48}$$

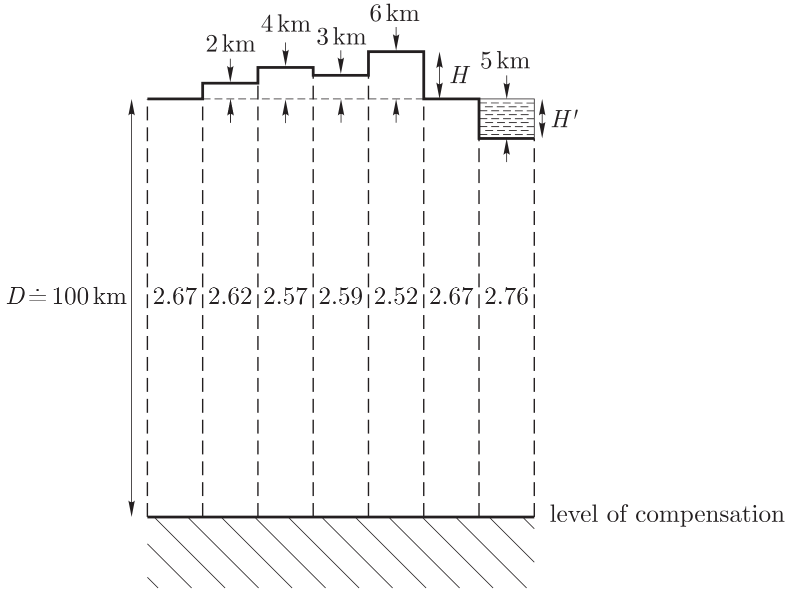

Fig. 3.9. Pratt–Hayford isostasy model

which expresses the condition of equal mass. It may be assumed that

$$\varrho_0 = 2.67 \text{ g cm}^{-3} . \tag{3–49}$$

According to (3–48), the actual density ϱ is slightly smaller than this normal value ϱ_0. Consequently, there is a mass deficiency which, according to (3–48), is given by

$$\Delta\varrho = \varrho_0 - \varrho = \frac{h}{D+H}\,\varrho_0 . \tag{3–50}$$

In the oceans, the condition of equal mass is expressed as

$$(D - H')\,\varrho + H'\,\varrho_w = D\,\varrho_0 , \tag{3–51}$$

where

$$\varrho_w = 1.027 \text{ g cm}^{-3} \tag{3–52}$$

is the density and H' the depth of the ocean. Hence, there is a mass surplus of a suboceanic column given by

$$\varrho - \varrho_0 = \frac{H'}{D - H'}\,(\varrho_0 - \varrho_w) . \tag{3–53}$$

As a matter of fact, this model of compensation is idealized and schematic. It can be only approximately fulfilled in nature. Values of the depth of compensation around

$$D = 100 \text{ km} \tag{3–54}$$

are assumed.

For a spheroidal earth, the columns will converge slightly towards its center, and other refinements may be introduced. We may postulate either equality of mass or equality of pressure; each postulate leads to somewhat different spherical refinements. It may be mentioned that for computational reasons Hayford used still another, slightly different model; for instance, he reckoned the depth of compensation D from the earth's surface instead of from sea level.

Airy–Heiskanen system

Airy proposed this model, and Heiskanen gave it a precise formulation for geodetic purposes and applied it extensively. Figure 3.10 illustrates the principle. The mountains of constant density

$$\varrho_0 = 2.67 \text{ g cm}^{-3} \tag{3–55}$$

float on a denser underlayer of constant density

$$\varrho_1 = 3.27 \text{ g cm}^{-3} . \tag{3–56}$$

The higher they are, the deeper they sink. Thus, root formations exist under mountains, and "antiroots" under the oceans.

We denote the density difference $\varrho_1 - \varrho_0$ by $\Delta\varrho$. On the basis of assumed numerical values, we have

$$\Delta\varrho = \varrho_1 - \varrho_0 = 0.6 \text{ g cm}^{-3} . \tag{3–57}$$

Denoting the height of the topography by H and the thickness of the corresponding root by t (Fig. 3.10), then the condition of floating equilibrium is

$$t\,\Delta\varrho = H\,\varrho_0 , \tag{3–58}$$

so that

$$t = \frac{\varrho_0}{\Delta\varrho} H = 4.45\ H \tag{3–59}$$

results. For the oceans, the corresponding condition is

$$t'\,\Delta\varrho = H'\,(\varrho_0 - \varrho_w) , \tag{3–60}$$

where H' and ϱ_w are defined as above and t' is the thickness of the antiroot (Fig. 3.10), so that we get

$$t' = \frac{\varrho_0 - \varrho_w}{\varrho_1 - \varrho_0} H' = 2.73\ H' \tag{3–61}$$

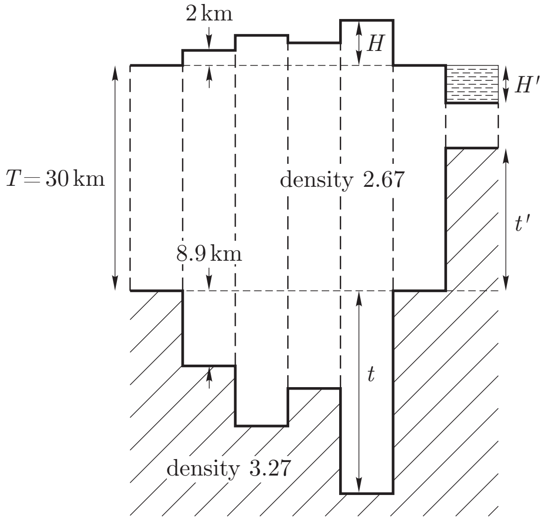

Fig. 3.10. Airy–Heiskanen isostasy model

for the numerical values assumed.

Again, spherical corrections must be applied to these formulas for higher accuracy, and the formulations in terms of equal mass and equal pressure lead to slightly different results.

The normal thickness of the earth's crust is denoted by T (Fig. 3.10); values of around

$$T = 30\,\mathrm{km} \tag{3–62}$$

are assumed. The crustal thickness under mountains is then

$$T + H + t \tag{3–63}$$

and under the oceans it is

$$T - H' - t' . \tag{3–64}$$

Vening Meinesz regional system

Both systems just discussed are highly idealized in that they assume the compensation to be strictly local; that is, they assume that compensation takes place along vertical columns. This presupposes free mobility of the masses to a degree that is obviously unrealistic in this strict form.

For this reason, Vening Meinesz modified the Airy floating theory in 1931, introducing regional instead of local compensation. The principal difference between these two kinds of compensation is illustrated by Fig. 3.11. In Vening

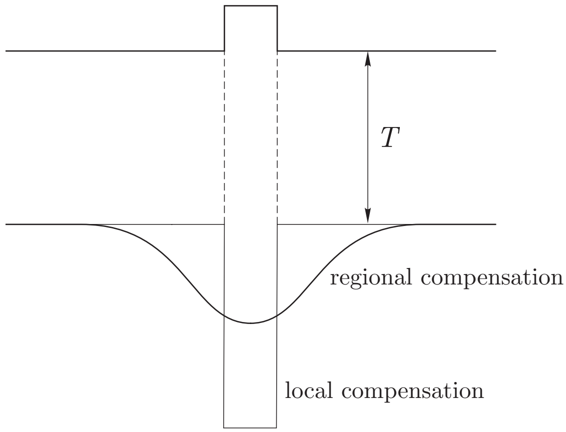

Fig. 3.11. Local and regional compensation

Meinesz' theory, the topography is considered as a load on an unbroken but yielding elastic crust.

In a very sloppy way which is only good for memorizing, we may say that, standing on thin ice, Airy will break through, but under Vening Meinesz the ice is stronger and will bend but not break.

Although Vening Meinesz' refinement of Airy's theory is more realistic, it is more complicated and is, therefore, seldom used by geodesists because, as we will see, any isostatic system, if consistently applied, serves for geodetic purposes as well.

Geophysical and geodetic evidence shows that the earth is about 90% isostatically compensated, but it is difficult to decide, at least from gravimetric evidence alone, which model best accounts for this compensation. Although seismic results indicate an Airy type of compensation, in some places the compensation seems to follow the Pratt model. Nature will never conform to any of these models to the degree of precision which we have assumed above. However, a well-defined and consistent mathematical formulation is certainly a necessary prerequisite for the application of isostasy for geodetic purposes.

For an extensive presentation of several types of isostasy, see Moritz (1990: Chap. 8). The Vening Meinesz model has been treated in detail by Abd-Elmotaal (1995); much information is also available in the internet. A classic on isostasy and its geophysical applications is Heiskanen and Vening Meinesz (1958: Chaps. 5 and 7).

3.6.2 Topographic-isostatic reductions

The objective of the topographic-isostatic reduction of gravity is the *regularization* of the earth's crust according to some model of isostasy. Regularization here means that we are trying to make the earth's crust as homogeneous as possible. The topographic masses are not completely removed as in the Bouguer reduction but are shifted into the interior of the geoid in order to make up the mass deficiencies that exist under the continents. In the topographic-isostatic model of Pratt and Hayford, the topographic masses are distributed between the level of compensation and sea level, in order to bring the crustal density from its original value to the constant standard value ϱ_0. In the Airy–Heiskanen model, the topographic masses are used to fill the roots of the continents, bringing the density from $\varrho_0 = 2.67\ \mathrm{g/cm^3}$ to $\varrho_1 = 3.27\ \mathrm{g/cm^3}$.

In other terms, the topography is removed together with its compensation, and the final result is ideally a homogeneous crust of density ϱ_0 and constant thickness D (Pratt–Hayford) or T (Airy–Heiskanen).

Thus we have three steps:

1. removal of topography,
2. removal of compensation,
3. free-air reduction to the geoid.

Steps 1 and 3 are known from Bouguer reduction, so that the techniques of Sect. 3.4 can be applied to them. Step 2 is new and will be discussed now for the two main topographic-isostatic systems.

Pratt–Hayford system

The method is the same as for the terrain correction, Sect. 3.4, Eq. (3–32). The attraction of the (negative) compensation is again computed by

$$A_C = \sum \Delta A\,, \tag{3–65}$$

where the attraction of a vertical column representing a compartment is given by (3–22) with

$$b = D\,, \qquad c = D + H_P \tag{3–66}$$

and ϱ replaced by the density defect $\Delta\varrho$. If the preceding Bouguer reduction were done with the original density ϱ of the column expressed by

$$\varrho = \frac{D}{D+H}\,\varrho_0 \tag{3–67}$$

according to (3–48), then $\Delta\varrho$ would be given by (3–50).

Usually the Bouguer reduction is performed using a constant density ϱ_0; the density defect $\Delta\varrho$ must then be computed by

$$\Delta\varrho = \frac{H}{D}\,\varrho_0\,, \tag{3–68}$$

which differs slightly from (3–50), in order to restore equality of mass according to

$$(\varrho_0 - \Delta\varrho)\,D + \varrho_0 H = \varrho_0 D\,. \tag{3–69}$$

The first term on the left-hand side represents the mass of the layer between the level of compensation and sea level; the second term represents the mass of the topography, now assumed to have a density ϱ_0.

Airy–Heiskanen system

Again we use

$$A_C = \sum \Delta A\,, \tag{3–70}$$

where b and c in (3–22) are, according to Fig. 3.12, given by

$$b = t\,, \qquad c = H_P + T + t\,, \tag{3–71}$$

and ϱ is replaced by $\Delta\varrho = \varrho_1 - \varrho_0 = 0.6$ g/cm^3.

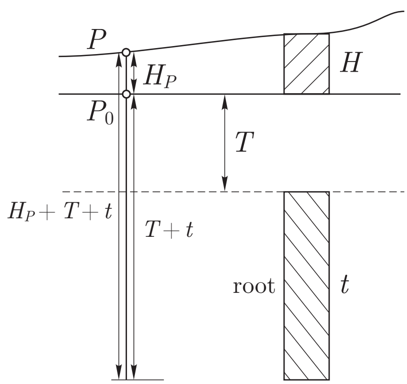

Fig. 3.12. Topography and compensation – Airy–Heiskanen model

Total reduction

In analogy with (3–38), the topographic-isostatically reduced gravity on the geoid becomes

$$g_{\mathrm{TI}} = g - A_T + A_C + F\,, \tag{3–72}$$

where $-A_C$ is the attraction of the compensation which is actually negative, so that its removal is equivalent to the term $+A_C$. The quantity A_T is the attraction of topography, to be computed as the effect of a Bouguer plate combined with terrain correction, Eq. (3–36), or in one step, as described in Sect. 3.4; F is the free-air reduction approximated by (3–26).

Oceanic stations

Here the terms A_T and F of (3–72) are zero, since the station is situated on the geoid, but the term A_C is more complicated.

In the Pratt–Hayford model, the procedure is as follows. The mass surplus (3–53) of a suboceanic column of height $D - H'$ (Fig. 3.9) is removed and used to fill the corresponding oceanic column of height H' to the proper density ϱ_0. In mathematical terms, this is

$$A_C = -A_1 + A_2 \,, \tag{3–73}$$

where both A_1 and A_2 are of the form (3–32), ΔA is given by (3–22). For A_1 we have

$$b = D - H' \,, \qquad c = D \,, \tag{3–74}$$

and density $\varrho - \varrho_0$; for A_2 we have

$$b = c = H' \tag{3–75}$$

and density $\varrho_0 - \varrho_w$.

In the Airy–Heiskanen model, the mass surplus of the antiroot, $\varrho_1 - \varrho_0$, is used to fill the oceans to the proper density ϱ_0. The corresponding value is again given by (3–73), where for A_1 we now have

$$b = t' \,, \qquad c = T \,, \tag{3–76}$$

and density $\varrho_1 - \varrho_0$; and for A_2 we have, as before,

$$b = c = H' \tag{3–77}$$

and density $\varrho_0 - \varrho_w$.

In both models, Eq. (3–72) reduces for oceanic stations to

$$g_{\mathrm{TI,\,ocean}} = g + A_C \,. \tag{3–78}$$

Topographic-isostatic anomalies

The topographic-isostatic gravity anomalies are – in analogy to the Bouguer anomalies – defined by

$$\Delta g_{\mathrm{TI}} = g_{\mathrm{TI}} - \gamma \,. \tag{3–79}$$

If any of the topographic-isostatic systems were rigorously true, then the topographic-isostatic reduction would fulfil perfectly its goal of complete regularization of the earth's crust, which would become level and homogeneous. Then, with a properly chosen reference model for γ, the topographic-isostatic gravity anomalies (3–79) would be zero.

The actual topographic-isostatic compensation occurring in nature cannot completely conform to such abstract models. As a consequence, nonzero topo-graphic-isostatic gravity anomalies will be left, but they will be small, smooth, and more or less randomly positive and negative. On account of this smoothness and independence of elevation, they are better suited for interpolation or extrapolation than any other type of anomalies; see Chap. 9, particularly Sect. 9.7.

It may be stressed again that for geodetic purposes the topographic-isostatic model used must be mathematically precise and self-consistent, and the same model must be used throughout. Refinements include the consideration of irregularities of density of the topographic masses and the consideration of the anomalous gradient of gravity.

3.7 The indirect effect

The removal or shifting of masses underlying the gravity reductions change the gravity potential and, hence, the geoid. This change of the geoid is an *indirect effect* of the gravity reductions.

Thus, the surface computed by Stokes' formula from topographic-isostatic gravity anomalies, is not the geoid itself but a slightly different surface, the cogeoid. To every gravity reduction there corresponds a different cogeoid.

Let the undulation of the cogeoid be N^c. Then the undulation N of the actual geoid is obtained from

$$N = N^c + \delta N \tag{3–80}$$

by taking into account the indirect effect on N, which is given by

$$\delta N = \frac{\delta W}{\gamma} \,, \tag{3–81}$$

where δW is the change of potential at the geoid. Equation (3–81) is an application of Bruns' theorem (2–237).

The change of potential, δW, is for the Bouguer reduction expressed by

$$\delta W_B = U_T \tag{3–82}$$

and for the topographic-isostatic reduction by

$$\delta W_{\mathrm{TI}} = U_T - U_C\,, \tag{3–83}$$

the subscripts of the potential U corresponding to those of the attraction A used in the preceding sections.

For the practical determination of U_T and U_C, the template technique, as expressed in (3–32), may again be used (at least, conceptually):

$$U = \sum \Delta U\,, \tag{3–84}$$

where the relevant formulas are the first equation of (3–21), (3–9), (3–12), and (3–15). The point U refers to is always the point P_0 at sea level (Fig. 3.1). For U_T we use U_0, see (3–12), with $b = H$ and density ϱ_0 (see Fig. 3.12). For U_C in the continental case, we use U_{e}, see (3–9), with the following values: Pratt–Hayford,

$$b = c = d\,, \qquad \text{density } \frac{H}{D}\,\varrho_0\,; \tag{3–85}$$

Airy–Heiskanen,

$$b = t\,, \qquad c = t + T\,, \qquad \text{density } \varrho_1 - \varrho_0\,. \tag{3–86}$$

The corresponding considerations for the oceanic case are left as an exercise for the reader.

The indirect effect with Bouguer anomalies is very large, of the order of ten times the geoidal undulation itself. See the map at the end of Helmert (1884: Tafel I), where the maximum value is 440 m! The reason is that the earth is in general topographic-isostatically compensated. Therefore, the Bouguer anomalies cannot be used for the determination of the geoid.

With topographic-isostatic gravity anomalies, as might be expected, the indirect effect is smaller than N, of the order of 10 m. It is necessary, however, to compute the indirect effect δN_I carefully, using exactly the same topographic-isostatic model as for the gravity reductions.

Furthermore, before applying Stokes' formula, the topographic-isostatic gravity anomalies must be reduced from the geoid to the cogeoid. This is done by a simple free-air reduction, using (3–26), by adding to Δg_I the correction

$$\delta = +0.3086\,\delta N \;[\mathrm{mgal}]\,, \tag{3–87}$$

δN in meters. This correction δ is the *indirect effect on gravity*; it is of the order of 3 mgal.

Now the topographic-isostatic gravity anomalies refer strictly to the cogeoid. The application of Stokes' formula gives N^c, which according to (3–80) is to be corrected by the indirect effect δN to give the undulation N of the actual geoid.

Deflections of the vertical

The indirect effect on the deflections of the vertical is, in agreement with Eqs. (2–377), given by

$$\begin{aligned} \delta\xi &= -\frac{1}{R}\frac{\partial\,\delta N}{\partial\varphi}\,, \\ \delta\eta &= -\frac{1}{R\cos\varphi}\frac{\partial\,\delta N}{\partial\lambda}\,. \end{aligned} \tag{3–88}$$

The indirect effect is essentially identical with the so-called topographic-isostatic deflection of the vertical (Heiskanen and Vening Meinesz 1958: pp. 252–255).

The *topographic-isostatic reduction* as such is very much alive, however. *It is practically the only gravity reduction used for geoid determination at the present time* (with the possible exception of free-air reduction, which is a case by itself).

The last purely gravimetric geoid, before the advent of satellites, was the Columbus Geoid (Heiskanen 1957).

3.8 The inversion reduction of Rudzki

It is possible to find a gravity reduction where the indirect effect is zero. This is done by shifting the topographic masses into the interior of the geoid in such a way that

$$U_C = U_T\,. \tag{3–89}$$

Then

$$\delta W = U_T - U_C = 0\,. \tag{3–90}$$

This procedure was given by M. P. Rudzki in 1905. For the present purpose, we may consider the geoid to be a sphere of radius R (Fig. 3.13). Let the mass element dm at Q be replaced by a mass element dm' at a certain point Q' inside the geoid situated on the same radius vector. The potential due to

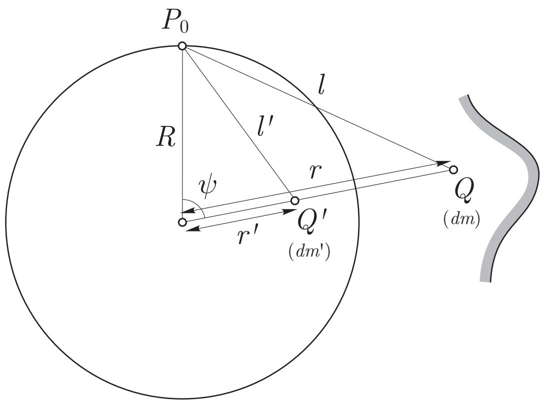

Fig. 3.13. Rudzki reduction as an inversion in a sphere

these mass elements at the geoidal point P_0 is

$$\begin{aligned} dU_T &= G\,\frac{dm}{l} = \frac{G\,dm}{\sqrt{r^2 + R^2 - 2R\,r\,\cos\psi}}\,, \\ dU_C &= G\,\frac{dm'}{l'} = \frac{G\,dm'}{\sqrt{r'^2 + R^2 - 2R\,r'\,\cos\psi}}\,. \end{aligned} \tag{3–91}$$

We should have

$$dU_C = dU_T \tag{3–92}$$

if

$$dm' = \frac{R}{r}\,dm \tag{3–93}$$

and

$$r' = \frac{R^2}{r}\,. \tag{3–94}$$

This is readily verified by substitution into the second equation of (3–91). The condition (3–94) means that Q' and Q are related by *inversion in the sphere* of radius R (Kellogg 1929: p. 231). Therefore, this reduction method is called *inversion reduction or Rudzki reduction.*

The condition (3–93) expresses the fact that the compensating mass dm' is not exactly equal to dm but is slightly smaller. Since this relative decrease of mass is of the order of 10^{-8}, it may be safely neglected by setting

$$dm' = dm\,. \tag{3–95}$$

Usually it is even sufficient to replace the sphere by a plane. Then Q' is the ordinary mirror image of Q (Fig. 3.14).

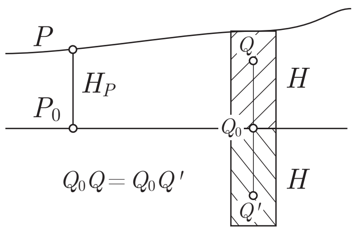

Fig. 3.14. Rudzki reduction as a plane approximation

Rudzki gravity at the geoid becomes, in analogy to (3–72),

$$g_R = g - A_T + A_C + F\,, \tag{3–96}$$

where $A_C = \sum \Delta A$ with $b = H$, $c = H + H_P$, the density being equal to that of topography.

Since the indirect effect is zero, the cogeoid of Rudzki coincides with the actual geoid, but the gravity field outside the earth is changed, which today is in the center of attention. In addition, the Rudzki reduction does not correspond to a geophysically meaningful model. *Nevertheless, it is important conceptually.* Regard it an interesting historic curiosity, but never even consider to use it!

3.9 The condensation reduction of Helmert

Here the topography is condensed so as to form a surface layer (somewhat like a glass sphere made of very thin but very heavy and robust glass) on the geoid so that the total mass remains unchanged. Again, the mass is shifted along the local vertical (Fig. 3.15).

We may consider Helmert's condensation as a limiting case of an isostatic

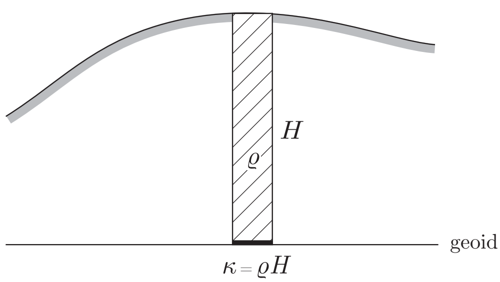

Fig. 3.15. Helmert's method of condensation

reduction according to the Pratt–Hayford system as the depth of compensation D goes to zero. This is sometimes useful.

Again we have

$$g_H = g - A_T + A_C + F\,, \tag{3–97}$$

where $A_C = \sum \Delta A$ is now to be computed using the second equation of (3–19) with $c = H_P$ and $\kappa = \varrho\, H$; H_P is the height of the station P and H the height of the compartment.

The indirect effect is

$$\delta W = U_T - U_C\,. \tag{3–98}$$

The potential $U_C = \sum \Delta U$ is to be computed using the first equation of (3–19) with $\kappa = \varrho\, H$ as before, but $c = 0$ since it refers to the geoidal point P_0. The corresponding δN is very small, amounting to about 1 m per 3 km of average elevation. It may, therefore, usually be neglected so that the cogeoid of the condensation reduction practically coincides with the actual geoid.

Even the "direct effect", $-A_T + A_C$, can usually be neglected, as the attraction of the Helmert layer nearly compensates that of the topography. There remains

$$g_H = g + F\,, \tag{3–99}$$

that is, the simple free-air reduction. In this sense, *the simple free-air reduction may be considered as giving approximate boundary values at the geoid*, to be used in Stokes' formula. To the same degree of approximation, the "free-air cogeoid" coincides with the actual geoid.

Hence, the free-air anomalies

$$\Delta g_F = g + F - \gamma \tag{3–100}$$

may be considered as approximations of "condensation anomalies"

$$\Delta g_H = g_H - \gamma\,. \tag{3–101}$$

The many facets of free-air reduction

This is one of the most basic, most difficult, and most fascinating topics of physical geodesy. In fact, the free-air anomaly means several conceptually different but related concepts.

1. The term F above has been seen to be *part of every gravity reduction* rather than a full-fledged gravity reduction itself.
2. Approximately, free-air anomalies may be identified with Helmert's condensation anomalies as we have seen above.

3. *Rigorously*, free-air anomalies can even be considered as resulting from a *mass-transporting* gravity reduction, in a similar sense as the isostatic anomaly. Just imagine that you transport the masses above the geoid into its interior in such a way that *the external potential remains unchanged*! This reminds us of Rudzki's reduction (geoid potential remains constant) but is rather different. The most important advantage is that the free-air anomaly in the present sense leaves the external potential unchanged which nowadays is much more important than the geoid. The greatest disavantage is that it cannot be computed: we do not know how to shift the masses so that the external masses remain unchanged. In logical terms, the Rudzki reduction is constructive – we are told how to do it –, whereas the present reduction is non-constructive – we do not know how to do it directly. More about this in Sects. 8.2, 8.6, 8.9, and 8.15. We shall, thus, attempt to cut the difficult cake into easier pieces.

These are the main methods that have been proposed for the reduction of gravity. A simple overview is given by Fig. 3.16.

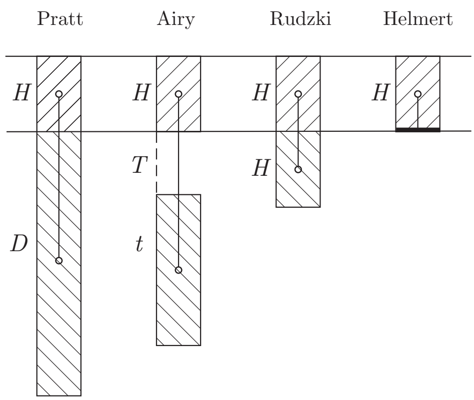

Fig. 3.16. Topography and compensation for different gravity reductions