2 Gravity field of the earth

2.1 Gravity

The total force acting on a body at rest on the earth's surface is the resultant of gravitational force and the centrifugal force of the earth's rotation and is called gravity.

Take a rectangular coordinate system whose origin is at the earth's center of gravity and whose z-axis coincides with the earth's mean axis of rotation (Fig. 2.1). The x- and y-axes are so chosen as to obtain a right-handed coordinate system; otherwise they are arbitrary. For convenience, we may assume an x-axis which is associated with the mean Greenwich meridian (it "points" towards the mean Greenwich meridian). Note that we are assuming in this book that the earth is a solid body rotating with constant speed around a fixed axis. This is a rather simplified assumption, see Moritz and Mueller (1987). The centrifugal force f on a unit mass is given by

$$f = \omega^2 p \,, \tag{2–1}$$

where ω is the angular velocity of the earth's rotation and

$$p = \sqrt{x^2 + y^2} \tag{2–2}$$

is the distance from the axis of rotation. The vector $\mathbf{f}$ of this force has the

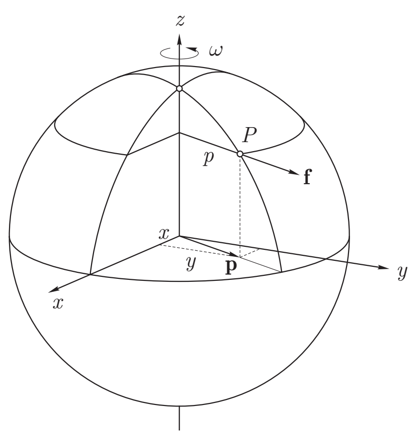

Fig. 2.1. The centrifugal force

direction of the vector

$$\mathbf{p} = [x,\, y,\, 0] \tag{2–3}$$

and is, therefore, given by

$$\mathbf{f} = \omega^2 \mathbf{p} = [\omega^2 x,\, \omega^2 y,\, 0]\,. \tag{2–4}$$

The centrifugal force can also be derived from a potential

$$\Phi = \frac{1}{2}\,\omega^2 (x^2 + y^2)\,, \tag{2–5}$$

so that

$$\mathbf{f} = \operatorname{grad} \Phi \equiv \left[\frac{\partial \Phi}{\partial x},\, \frac{\partial \Phi}{\partial y},\, \frac{\partial \Phi}{\partial z}\right]. \tag{2–6}$$

Substituting (2–5) into (2–6) yields (2–4).

In the introductory remark above, we mentioned that gravity is the resultant of gravitational force and centrifugal force. Accordingly, the potential of gravity, W, is the sum of the potentials of gravitational force, V, cf. (1–12), and centrifugal force, Φ:

$$W = W(x,\, y,\, z) = V + \Phi = G \iiint\limits_{v} \frac{\varrho}{l}\, dv + \frac{1}{2}\,\omega^2 (x^2 + y^2)\,, \tag{2–7}$$

where the integration is extended over the earth.

Differentiating (2–5), we find

$$\Delta \Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 2\omega^2\,. \tag{2–8}$$

If we combine this with Poisson's equation (1–17) for V, we get the *generalized Poisson equation* for the gravity potential W:

$$\Delta W = -4\pi\, G\, \varrho + 2\omega^2\,. \tag{2–9}$$

Since Φ is an analytic function, the discontinuities of W are those of V: some second derivatives have jumps at discontinuities of density.

The gradient vector of W,

$$\mathbf{g} = \operatorname{grad} W \equiv \left[\frac{\partial W}{\partial x},\, \frac{\partial W}{\partial y},\, \frac{\partial W}{\partial z}\right] \tag{2–10}$$

with components

$$g_x = \frac{\partial W}{\partial x} = -G \iiint_v \frac{x-\xi}{l^3}\,\varrho\,dv + \omega^2 x\,,$$
$$g_y = \frac{\partial W}{\partial y} = -G \iiint_v \frac{y-\eta}{l^3}\,\varrho\,dv + \omega^2 y\,, \tag{2–11}$$
$$g_z = \frac{\partial W}{\partial z} = -G \iiint_v \frac{z-\zeta}{l^3}\,\varrho\,dv\,,$$

is called the *gravity vector*; it is the total force (gravitational force plus centrifugal force) acting on a unit mass. As a vector, it has *magnitude* and *direction.*

The magnitude g is called gravity in the narrower sense. It has the physical dimension of an acceleration and is measured in gal (1 gal = 1 cm s^{-2}), the unit being named in honor of Galileo Galilei. The numerical value of g is about 978 gal at the equator, and 983 gal at the poles. In geodesy, another unit is often convenient – the milligal, abbreviated mgal (1 mgal = 10^{-3} gal).

In SI units, we have

$$\begin{aligned} 1\text{ gal} &= 0.01\ \text{m}\,\text{s}^{-2}\,, \\ 1\text{ mgal} &= 10\,\mu\text{m}\,\text{s}^{-2}\,. \end{aligned} \tag{2–12}$$

The direction of the gravity vector is the direction of the *plumb line*, or the vertical; its basic significance for geodetic and astronomical measurements is well known.

In addition to the centrifugal force, another force called the *Coriolis force* acts on a moving body. It is proportional to the velocity with respect to the earth, so that it is zero for bodies resting on the earth. Since in classical geodesy (i.e., not considering navigation) we usually deal with instruments at rest relative to the earth, the Coriolis force plays no role here and need not be considered.

Gravitational and inertial mass

The reader may have noticed that the mass m has been used in two conceptually completely different senses: as *inertial mass* in Newton's law of inertia, *force* = *mass* × *acceleration* and as *gravitational mass* in Newton's law of gravitation (1–1). Thus, m in gravitation, which is a "true" force, is the gravitational mass, but m in the centrifugal "force", which is an acceleration, is the inertial mass. The Hungarian physicist Roland Eötvös had shown experimentally already around 1890 that both kinds of masses are equal within 10^{-11}, which is a formidable accuracy. He used the same type of instrument by which experimental physicists have been able to determine the numerical value of the gravitational constant G only to a poor accuracy of about 10^{-4}, as we have seen at the beginning of this book. The coincidence between the inertial and the gravitational mass was far too good to be a physical accident, but, within classical mechanics, it was an inexplicable miracle. It was not before 1915 that Einstein made it one of the pillars of the general theory of relativity!

2.2 Level surfaces and plumb lines

The surfaces

$$W(x, y, z) = \text{constant}\,, \qquad (2\text{–}13)$$

on which the potential W is constant, are called *equipotential surfaces* or *level surfaces.*

Differentiating the gravity potential $W = W(x, y, z)$, we find

$$dW = \frac{\partial W}{\partial x}\,dx + \frac{\partial W}{\partial y}\,dy + \frac{\partial W}{\partial z}\,dz\,. \qquad (2\text{–}14)$$

In vector notation, using the scalar product, this reads

$$dW = \operatorname{grad} W \cdot d\mathbf{x} = \mathbf{g} \cdot d\mathbf{x}\,, \qquad (2\text{–}15)$$

where

$$d\mathbf{x} = [dx,\, dy,\, dz]\,. \qquad (2\text{–}16)$$

If the vector $d\mathbf{x}$ is taken along the equipotential surface $W =$ constant, then the potential remains constant and $dW = 0$, so that (2–15) becomes

$$\mathbf{g} \cdot d\mathbf{x} = 0\,. \qquad (2\text{–}17)$$

If the scalar product of two vectors is zero, then these vectors are orthogonal to each other. This equation therefore expresses the well-known fact that the *gravity vector is orthogonal to the equipotential surface* passing through the same point.

The surface of the oceans, after some slight idealization, is part of a certain level surface. This particular equipotential surface was proposed as the "mathematical figure of the earth" by C.F. Gauss, the "Prince of Mathematicians", and was later termed the *geoid.* This definition has proved highly suitable, and the geoid is still frequently considered by many to be the fundamental surface of physical geodesy. The geoid is thus defined by

$$W = W_0 = \text{constant}\,. \qquad (2\text{–}18)$$

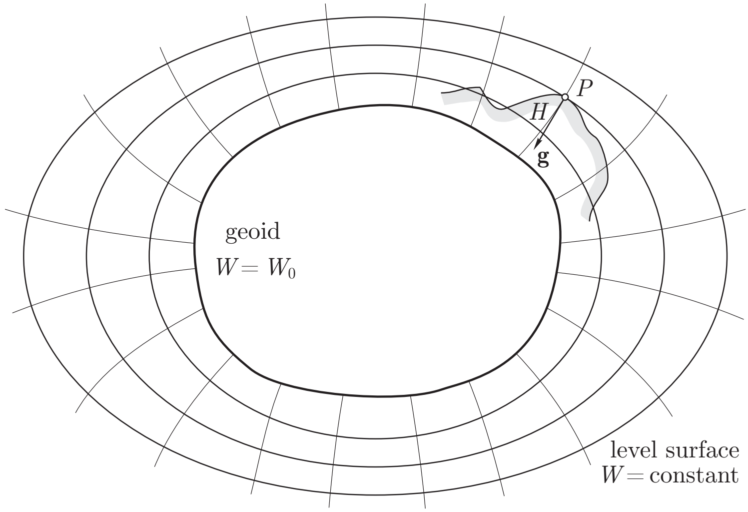

Fig. 2.2. Level surfaces and plumb lines

If we look at equation (2–7) for the gravity potential W, we can see that the equipotential surfaces, expressed by $W(x, y, z) =$ constant, are rather complicated mathematically. The level surfaces that lie completely outside the earth are at least analytical surfaces, although they have no *simple* analytical expression, because the gravity potential W is analytical outside the earth. This is not true of level surfaces that are partly or wholly inside the earth, such as the geoid. They are continuous and "smooth" (i.e., without edges), but they are no longer analytical surfaces; we will see in the next section that the curvature of the interior level surfaces changes discontinuously with the density.

The lines that intersect all equipotential surfaces orthogonally are not exactly straight but slightly curved (Fig. 2.2). They are called *lines of force*, or *plumb lines.* The gravity vector at any point is tangent to the plumb line at that point, hence "direction of the gravity vector", "vertical", and "direction of the plumb line" are synonymous. Sometimes this direction itself is briefly denoted as "plumb line".

As the level surfaces are, so to speak, horizontal everywhere, they share the strong intuitive and physical significance of the horizontal; and they share the geodetic importance of the plumb line because they are orthogonal to it. Thus, we understand why so much attention is paid to the equipotential surfaces.

The height H of a point above sea level (also called the *orthometric height*) is measured along the curved plumb line, starting from the geoid

(Fig. 2.2). If we take the vector $d\mathbf{x}$ along the plumb line, in the direction of increasing height H, then its length will be

$$\|d\mathbf{x}\| = dH \tag{2–19}$$

and its direction is opposite to the gravity vector $\mathbf{g}$, which points downward, so that the angle between $d\mathbf{x}$ and $\mathbf{g}$ is 180°. Using the definition of the scalar product (i.e., for two vectors $\mathbf{a}$ and $\mathbf{b}$ it is defined as $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\omega$, where ω is the angle between the two vectors), we get

$$\mathbf{g} \cdot d\mathbf{x} = g\, dH \cos 180^\circ = -g\, dH \tag{2–20}$$

accordingly, so that Eq. (2–15) becomes

$$dW = -g\, dH\,. \tag{2–21}$$

This equation relates the height H to the potential W and will be basic for the theory of height determination (Chap. 4). It shows clearly the inseparable interrelation that characterizes geodesy – the interrelation between the geometrical concepts (H) and the dynamic concepts (W).

Another form of Eq. (2–21) is

$$g = -\frac{\partial W}{\partial H}\,. \tag{2–22}$$

It shows that gravity is the negative *vertical gradient* of the potential W, or the negative vertical component of the gradient vector grad W.

Since geodetic measurements (theodolite measurements, leveling, but also satellite techniques etc.) are almost exclusively referred to the system of level surfaces and plumb lines, the geoid plays an essential part. Thus, we see why the aim of physical geodesy has been formulated as the *determination of the level surfaces of the earth's gravity field*. In a still more abstract but equivalent formulation, we may also say that physical geodesy aims at the determination of the potential function $W(x, y, z)$. At a first glance, the reader is probably perplexed about this definition, which is due to Bruns (1878), but its meaning is easily understood: If the potential W is given as a function of the coordinates x, y, z, then we know all level surfaces including the geoid; they are given by the equation

$$W(x,\, y,\, z) = \text{constant.} \tag{2–23}$$

2.3 Curvature of level surfaces and plumb lines

The formula for the curvature of a curve $y = f(x)$ is

$$\kappa = \frac{1}{\varrho} = \frac{y''}{(1 + y'^2)^{3/2}}\,, \tag{2–24}$$

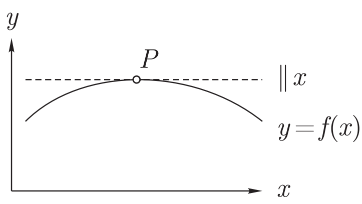

Fig. 2.3. The curvature of a curve

where κ is the curvature, ϱ is the radius of curvature, and

$$y' = \frac{dy}{dx}\,, \quad y'' = \frac{d^2y}{dx^2}\,. \tag{2–25}$$

If we use a plane local coordinate system xy in which a parallel to the x-axis is tangent at the point P under consideration (Fig. 2.3), then this implies $y' = 0$ and we get simply

$$\kappa = \frac{1}{\varrho} = \frac{d^2y}{dx^2}\,. \tag{2–26}$$

Level surfaces

Consider now a point P on a level surface S. Take a local coordinate system xyz with origin at P whose z-axis is vertical, that is, orthogonal to the surface S (Fig. 2.4). We intersect this level surface

$$W(x,\, y,\, z) = \text{constant} \tag{2–27}$$

with the xz-plane by setting

$$y = 0\,. \tag{2–28}$$

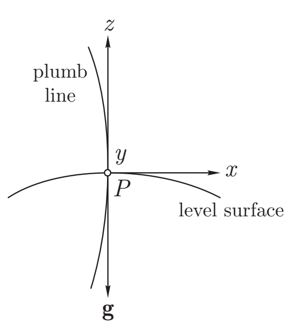

Fig. 2.4. The local coordinate system

Comparing Fig. 2.4 with Fig. 2.3, we see that z now takes the place of y. Therefore, instead of (2–26) we have for the curvature of the intersection of the level surface with the xz-plane:

$$K_1 = \frac{d^2 z}{dx^2} . \tag{2–29}$$

If we differentiate $W(x, y, z) = W_0$ with respect to x, considering that y is zero and z is a function of x, we get

$$\begin{gathered} W_x + W_z \frac{dz}{dx} = 0 , \\ W_{xx} + 2W_{xz} \frac{dz}{dx} + W_{zz} \left(\frac{dz}{dx}\right)^2 + W_z \frac{d^2 z}{dx^2} = 0 , \end{gathered} \tag{2–30}$$

where the subscripts denote partial differentiation:

$$W_x = \frac{\partial W}{\partial x} , \quad W_{xz} = \frac{\partial^2 W}{\partial x \, \partial z} , \quad \ldots . \tag{2–31}$$

Since the x-axis is tangent at P, we get $dz/dx = 0$ at P, so that

$$\frac{d^2 z}{dx^2} = -\frac{W_{xx}}{W_z} . \tag{2–32}$$

Since the z-axis is vertical, we have, using (2–22),

$$W_z = \frac{\partial W}{\partial z} = \frac{\partial W}{\partial H} = -g . \tag{2–33}$$

Therefore, Eq. (2–29) becomes

$$K_1 = \frac{W_{xx}}{g} . \tag{2–34}$$

The curvature of the intersection of the level surface with the yz-plane is found by replacing x with y:

$$K_2 = \frac{W_{yy}}{g} . \tag{2–35}$$

The mean curvature J of a surface at a point P is defined as the arithmetic mean of the curvatures of the curves in which two mutually perpendicular planes through the surface normal intersect the surface (Fig. 2.5). Hence, we find

$$J = -\frac{1}{2}(K_1 + K_2) = -\frac{W_{xx} + W_{yy}}{2g} . \tag{2–36}$$

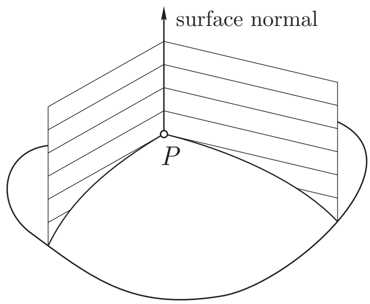

Fig. 2.5. Definition of mean curvature

Here the minus sign is only a convention. This is an expression for the *mean curvature of the level surface.*

From the generalized Poisson equation

$$\Delta W \equiv W_{xx} + W_{yy} + W_{zz} = -4\pi\, G\, \varrho + 2\omega^2\,, \qquad (2\text{–}37)$$

we find

$$-2g\, J + W_{zz} = -4\pi\, G\, \varrho + 2\omega^2\,. \qquad (2\text{–}38)$$

Considering

$$W_z = -g\,, \quad W_{zz} = -\frac{\partial g}{\partial z} = -\frac{\partial g}{\partial H}\,, \qquad (2\text{–}39)$$

we finally obtain

$$\frac{\partial g}{\partial H} = -2g\, J + 4\pi\, G\, \varrho - 2\omega^2\,. \qquad (2\text{–}40)$$

This important equation, relating the *vertical gradient of gravity* $\partial g/\partial H$ to the mean curvature of the level surface, is also due to Bruns (1878). It is another beautiful example of the interrelation between the geometric and dynamic concepts in geodesy.

Plumb lines

The curvature of the plumb line is needed for the reduction of astronomical observations to the geoid. A plumb line may be defined as a curve whose line element vector

$$d\mathbf{x} = [dx,\, dy,\, dz] \qquad (2\text{–}41)$$

has the direction of the gravity vector

$$\mathbf{g} = [W_x,\, W_y,\, W_z]\,; \qquad (2\text{–}42)$$

that is, $d\mathbf{x}$ and $\mathbf{g}$ differ only by a proportionality factor. This is best expressed in the form

$$\frac{dx}{W_x} = \frac{dy}{W_y} = \frac{dz}{W_z}\,. \tag{2–43}$$

In the coordinate system of Fig. 2.4, the curvature of the projection of the plumb line onto the xz-plane is given by

$$\kappa_1 = \frac{d^2x}{dz^2}\,; \tag{2–44}$$

this is equation (2–26) applied to the present case. Using (2–43), we have

$$\frac{dx}{dz} = \frac{W_x}{W_z}\,. \tag{2–45}$$

We differentiate with respect to z, considering that $y = 0$:

$$\frac{d^2x}{dz^2} = \frac{1}{W_z^2}\left[W_z\left(W_{xz} + W_{xx}\,\frac{dx}{dz}\right) - W_x\left(W_{zz} + W_{zx}\,\frac{dx}{dz}\right)\right]. \tag{2–46}$$

In our particular coordinate system, the gravity vector coincides with the z-axis, so that its x- and y-components are zero:

$$W_x = W_y = 0\,. \tag{2–47}$$

Figure 2.4 shows that we also have

$$\frac{dx}{dz} = 0\,. \tag{2–48}$$

Therefore,

$$\frac{d^2x}{dz^2} = \frac{W_z\,W_{xz}}{W_z^2} = \frac{W_{xz}}{W_z} = \frac{W_{zx}}{W_z}\,. \tag{2–49}$$

Considering $W_z = -g$, we finally obtain

$$\kappa_1 = \frac{1}{g}\,\frac{\partial g}{\partial x} \tag{2–50}$$

and, similarly,

$$\kappa_2 = \frac{1}{g}\,\frac{\partial g}{\partial y}\,. \tag{2–51}$$

These are the curvatures of the projections of the plumb line onto the xz- and yz-plane, the z-axis being vertical, that is, coinciding with the gravity vector. The total curvature κ of the plumb line is given, according to differential geometry (essentially Pythagoras' theorem), by

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2} = \frac{1}{g}\sqrt{g_x^2 + g_y^2}\,. \tag{2–52}$$

For reducing astronomical observations (Sect. 5.12), we need only the projection curvatures (2–50) and (2–51).

We mention finally that the various formulas for the curvature of level surfaces and plumb lines are equivalent to the single vector equation

$$\operatorname{grad} g = (-2g\,J + 4\pi\,G\,\varrho - 2\omega^2)\,\mathbf{n} + g\,\kappa\,\mathbf{n}_1\,, \tag{2–53}$$

where $\mathbf{n}$ is the unit vector along the plumb line (its unit tangent vector) and $\mathbf{n}_1$ is the unit vector along the principal normal to the plumb line. This may be easily verified. Using the local xyz-system, we have

$$\begin{aligned} \mathbf{n} &= [0,\, 0,\, 1]\,, \\ \mathbf{n}_1 &= [\cos\alpha,\, \sin\alpha,\, 0]\,, \end{aligned} \tag{2–54}$$

where α is the angle between the principal normal and the x-axis (Fig. 2.6). The z-component of (2–53) yields Bruns' equation (2–40), and the horizontal components yield

$$\frac{\partial g}{\partial x} = g\,\kappa\,\cos\alpha\,, \quad \frac{\partial g}{\partial y} = g\,\kappa\,\sin\alpha\,. \tag{2–55}$$

These are identical to (2–50) and (2–51), since $\kappa_1 = \kappa\,\cos\alpha$ and $\kappa_2 = \kappa\,\sin\alpha$, as differential geometry shows. Equation (2–53) is called the *generalized Bruns equation.*

More about the curvature properties and the "inner geometry" of the gravitational field will be found in books by, e.g., Hotine (1969: Chaps. 4–20), Marussi (1985) and Moritz and Hofmann-Wellenhof (1993: Chap. 3).

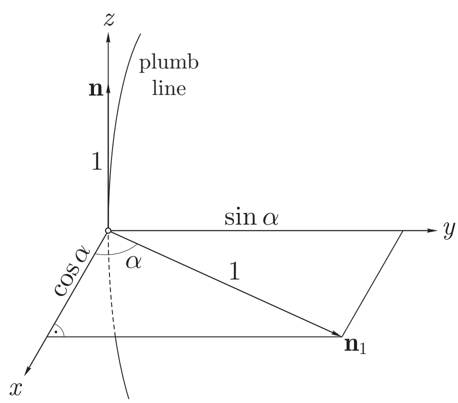

Fig. 2.6. Generalized Bruns equation

2.4 Natural coordinates

The system of level surfaces and plumb lines may be used as a three-dimensional curvilinear coordinate system that is well suited to certain purposes; these coordinates can be measured directly, as opposed to local rectangular coordinates x, y, z. Note, however, that global rectangular coordinates may be measured directly using satellites, see Sect. 5.3.

The direction of the earth's axis of rotation and the position of the equatorial plane (normal to the axis) are well defined astronomically. The *astronomical latitude* Φ of a point P is the angle between the vertical (direction of the plumb line) at P and the equatorial plane, see Fig. 2.7. From this figure, we also see that line PN is parallel to the rotation axis, plane GPF normal to it, that is, parallel to the equatorial plane; $\mathbf{n}$ is the unit vector along the plumb line; plane NPF is the meridian plane of P, and plane NPG is parallel to the meridian plane of Greenwich.

Consider now a straight line through P parallel to the earth's axis of rotation. This parallel and the vertical at P together define the meridian plane of P. The angle between this meridian plane and the meridian plane of Greenwich (or some other fixed plane) is the *astronomical longitude* Λ of P. (Exercise: define Φ and Λ without using the unit sphere. The solution may be found in Sect. 5.9).

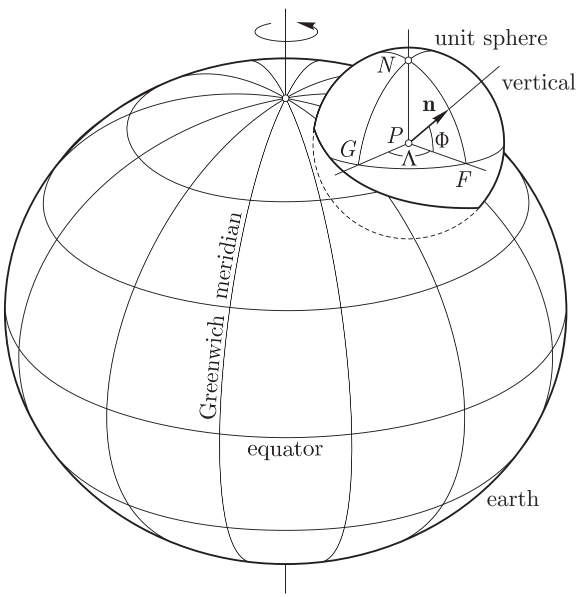

Fig. 2.7. Definition of the astronomical coordinates Φ and Λ of P by means of a unit sphere with center at P

The astronomical coordinates, latitude Φ and longitude Λ, form two of the three spatial coordinates of P. As third coordinate we may take the orthometric height H of P or its potential W. Equivalent to W is the *geopotential number* $C = W_0 - W$, where W_0 is the potential of the geoid. The orthometric height H was defined in Sect. 2.2; see also Fig. 2.2. The relations between W, C, and H are given by the equations

$$\begin{aligned} W &= W_0 - \int_0^H g\,dH = W_0 - C\,, \\ C &= W_0 - W = \int_0^H g\;dH\,, \\ H &= -\int_{W_0}^W \frac{dW}{g} = \int_0^C \frac{dC}{g}\,, \end{aligned} \tag{2–56}$$

which follow from integrating (2–21). The integral is taken along the plumb line of point P, starting from the geoid, where $H = 0$ and $W = W_0$ (see also Fig. 2.8).

The quantities

$$\Phi,\ \Lambda,\ W \quad \text{or} \quad \Phi,\ \Lambda,\ H \tag{2–57}$$

are called *natural coordinates*. They are the real-earth counterparts of the ellipsoidal coordinates. They are related in the following way to the geocentric rectangular coordinates x, y, z of Sect. 2.1. The x-axis is associated with the mean Greenwich meridian; from Fig. 2.7 we read that the unit vector of the vertical $\mathbf{n}$ has the xyz-components

$$\mathbf{n} = [\cos\Phi\ \cos\Lambda,\ \cos\Phi\ \sin\Lambda,\ \sin\Phi]\,; \tag{2–58}$$

the gravity vector $\mathbf{g}$ is known to be

$$\mathbf{g} = [W_x,\ W_y,\ W_z]\,. \tag{2–59}$$

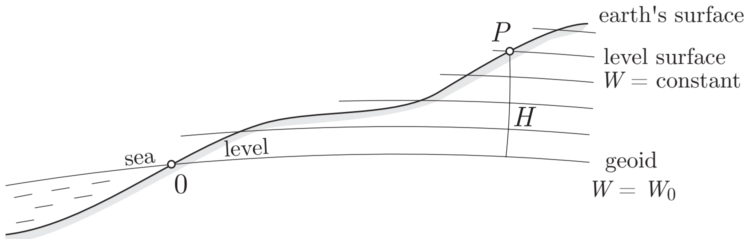

Fig. 2.8. The orthometric height H

On the other hand, since $\mathbf{n}$ is the unit vector corresponding to $\mathbf{g}$ but of opposite direction, it is given by

$$\mathbf{n} = -\frac{\mathbf{g}}{\|\mathbf{g}\|} = -\frac{\mathbf{g}}{g}\,, \tag{2–60}$$

so that

$$\mathbf{g} = -g\,\mathbf{n}\,. \tag{2–61}$$

This equation, together with (2–58) and (2–59), gives

$$\begin{aligned} -W_x &= g\,\cos\Phi\,\cos\Lambda\,,\\ -W_y &= g\,\cos\Phi\,\sin\Lambda\,,\\ -W_z &= g\,\sin\Phi\,. \end{aligned} \tag{2–62}$$

Solving for Φ and Λ, we finally obtain

$$\begin{aligned} \Phi &= \tan^{-1}\frac{-W_z}{\sqrt{W_x^2+W_y^2}}\,,\\ \Lambda &= \tan^{-1}\frac{W_y}{W_x}\,,\\ W &= W(x,\,y,\,z)\,. \end{aligned} \tag{2–63}$$

These three equations relate the natural coordinates Φ, Λ, W to the rectangular coordinates x, y, z, provided the function $W = W(x, y, z)$ is known.

We see that Φ, Λ, H are related to x, y, z in a considerably more complicated way than the spherical coordinates r, ϑ, λ of Sect. 1.4. Note also the conceptual difference between the astronomical longitude Λ and the geocentric longitude λ.

2.5 The potential of the earth in terms of spherical harmonics

Looking at the expression (2–7) for the gravity potential W, we see that the part most difficult to handle is the gravitational potential V, the centrifugal potential being a simple analytic function.

The gravitational potential V can be made more manageable for many purposes if we keep in mind the fact that outside the attracting masses it is a harmonic function and can therefore be expanded into a series of spherical harmonics.

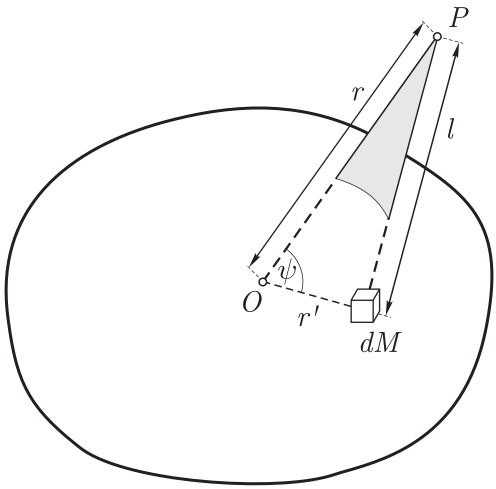

Fig. 2.9. Expansion into spherical harmonics

We now evaluate the coefficients of this series. The gravitational potential V is given by the basic equation (1–12):

$$V = G \iiint_{\text{earth}} \frac{dM}{l} \,, \tag{2–64}$$

where we now denote the mass element by dM; the integral is extended over the entire earth. Into this integral we substitute the expression (1–104):

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \,, \tag{2–65}$$

where the P_n are the conventional Legendre polynomials, r is the radius vector of the fixed point P at which V is to be determined, r' is the radius vector of the variable mass element dM, and ψ is the angle between r and r' (Fig. 2.9).

Since r is a constant with respect to the integration over the earth, it can be taken out of the integral. Thus, we get

$$V = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} G \iiint_{\text{earth}} r'^n P_n(\cos\psi)\, dM \,. \tag{2–66}$$

Writing this in the usual form as a series of solid spherical harmonics,

$$V = \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \lambda)}{r^{n+1}} \,, \tag{2–67}$$

we see by comparison that the Laplace surface spherical harmonic $Y_n(\vartheta, \lambda)$ is given by

$$Y_n(\vartheta, \lambda) = G \iiint_{\text{earth}} r'^n \, P_n(\cos\psi) \, dM \,, \tag{2–68}$$

the dependence on ϑ and λ arises from the angle ψ since

$$\cos\psi = \cos\vartheta \, \cos\vartheta' + \sin\vartheta \, \sin\vartheta' \cos(\lambda' - \lambda) \,. \tag{2–69}$$

The spherical coordinates ϑ, λ have been defined in Sect. 1.4.

A more explicit form is obtained by using the decomposition formula (1–108):

$$\frac{1}{l} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{2n+1} \left[\frac{\bar{\mathcal{R}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \, r'^n \bar{\mathcal{R}}_{nm}(\vartheta', \lambda') + \frac{\bar{\mathcal{S}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \, r'^n \bar{\mathcal{S}}_{nm}(\vartheta', \lambda') \right] . \tag{2–70}$$

Substituting this relation into the integral (2–64), we obtain

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\bar{A}_{nm} \, \frac{\bar{\mathcal{R}}_{nm}(\vartheta, \lambda)}{r^{n+1}} + \bar{B}_{nm} \, \frac{\bar{\mathcal{S}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \right] , \tag{2–71}$$

where the constant coefficients $\bar{A}_{nm}$ and $\bar{B}_{nm}$ are given by

$$\begin{aligned} (2n+1) \, \bar{A}_{nm} &= G \iiint_{\text{earth}} r'^n \, \bar{\mathcal{R}}_{nm}(\vartheta', \lambda') \, dM \,, \\ (2n+1) \, \bar{B}_{nm} &= G \iiint_{\text{earth}} r'^n \, \bar{\mathcal{S}}_{nm}(\vartheta', \lambda') \, dM \,. \end{aligned} \tag{2–72}$$

These formulas are very symmetrical and easy to remember: the coefficient, multiplied by $2n+1$, of the solid harmonic

$$\frac{\bar{\mathcal{R}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \tag{2–73}$$

is the integral of the solid harmonic

$$r'^n \bar{\mathcal{R}}_{nm}(\vartheta', \lambda') \,. \tag{2–74}$$

An analogous relation results for $\bar{\mathcal{S}}_{nm}$.

Note the nice analogy: V is a *sum* and the coefficients are *integrals*!

Since the mass element is

$$dM = \varrho \, dx' \, dy' \, dz' = \varrho \, r'^2 \sin\vartheta' \, dr' \, d\vartheta' \, d\lambda' \,, \tag{2–75}$$

the actual evaluation of the integrals requires that the density ϱ be expressed as a function of r', ϑ', λ'. Although no such expression is available at present, this fact does not diminish the theoretical and practical significance of spherical harmonics, since the coefficients A_{nm}, B_{nm} can be determined from the boundary values of gravity at the earth's surface. This is a boundary-value problem (see Sect. 1.13) and will be elaborated later.

Recalling the relations (1–91) and (1–98) between conventional and fully normalized spherical harmonics, we can also write equations (2–71) and (2–72) in terms of conventional harmonics, readily obtaining

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[A_{nm} \frac{\mathcal{R}_{nm}(\vartheta, \lambda)}{r^{n+1}} + B_{nm} \frac{\mathcal{S}_{nm}(\vartheta, \lambda)}{r^{n+1}} \right], \tag{2–76}$$

where

$$\begin{aligned} A_{n0} &= G \iiint_{\text{earth}} r'^n \, P_n(\cos\vartheta') \, dM \, ; \\ A_{nm} &= 2 \frac{(n-m)!}{(n+m)!} G \iiint_{\text{earth}} r'^n \, \mathcal{R}_{nm}(\vartheta', \lambda') \, dM \\ B_{nm} &= 2 \frac{(n-m)!}{(n+m)!} G \iiint_{\text{earth}} r'^n \, \mathcal{S}_{nm}(\vartheta', \lambda') \, dM \end{aligned} \Bigg\} \quad (m \neq 0) \, . \tag{2–77}$$

In connection with satellite dynamics, the potential V is often written in the form

$$V = \frac{GM}{r} \left\{ 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r} \right)^n \left[C_{nm} \, \mathcal{R}_{nm}(\vartheta, \lambda) + S_{nm} \, \mathcal{S}_{nm}(\vartheta, \lambda) \right] \right\}, \tag{2–78}$$

where a is the equatorial radius of the earth, so that

$$\left. \begin{aligned} A_{nm} &= GM \, a^n \, C_{nm} \\ B_{nm} &= GM \, a^n \, S_{nm} \end{aligned} \right\} \quad (n \neq 0) \, . \tag{2–79}$$

Distinguish the coefficient S_{nm} and the function $\mathcal{S}_{nm}$! The coefficient C_{n0} has formerly been denoted by $-J_n$. Note that C is related to cosine and S is related to sine.

The corresponding fully normalized coefficients

$$\bar{C}_{n0} = \frac{1}{\sqrt{2n+1}} C_{n0} ,$$

$$\left.\begin{aligned} \bar{C}_{nm} &= \sqrt{\frac{(n+m)!}{2(2n+1)(n-m)!}}\, C_{nm} \\ \bar{S}_{nm} &= \sqrt{\frac{(n+m)!}{2(2n+1)(n-m)!}}\, S_{nm} \end{aligned}\right\} \quad (m \neq 0) \tag{2–80}$$

are also used.

It is obvious that the nonzonal terms ($m \neq 0$) would be missing in all these expansions if the earth had complete rotational symmetry, since the terms mentioned depend on the longitude λ. In rotationally symmetrical bodies there is no dependence on λ because all longitudes are equivalent. The tesseral and sectorial harmonics will be small, however, since the deviations from rotational symmetry are slight.

Finally, we discuss the convergence of (2–71), or of the equivalent series expansions, of the earth's potential. This series is an expansion in powers of $1/r$. Therefore, the larger r is, the better the convergence. For smaller r it is not necessarily convergent. For an arbitrary body, the expansion of V into spherical harmonics can be shown to converge always outside the smallest sphere $r = r_0$ that completely encloses the body (Fig. 2.10). Inside this sphere, the series is usually divergent. In certain cases it can converge partly inside the sphere $r = r_0$. If the earth were a homogeneous ellipsoid of about

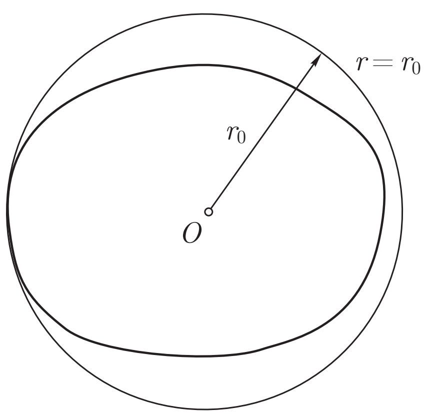

Fig. 2.10. Spherical-harmonic expansion of V converges outside the sphere $r = r_0$

the same dimensions, then the series for V would indeed still converge at the surface of the earth. Owing to the mass irregularities, however, the series of the actual potential V of the earth can be divergent or also convergent at the surface of the earth. *Theoretically*, this makes the use of a harmonic expansion of V at the earth's surface somewhat difficult; *practically*, it is always safe to regard it as convergent. For a detailed discussion see Moritz (1980 a: Sects. 6 and 7) and Sect. 8.6 herein.

It need hardly be pointed out that the spherical-harmonic expansion, always expressing a harmonic function, can represent only the potential outside the attracting masses, never inside.

2.6 Harmonics of lower degree

It is instructive to evaluate the coefficients of the first few spherical harmonics explicitly. For ready reference, we first state some conventional harmonic functions $\mathcal{R}_{nm}$ and $\mathcal{S}_{nm}$, using (1–60), (1–66), and (1–82):

$$\begin{aligned}
\mathcal{R}_{00} &= 1\,, & \mathcal{S}_{00} &= 0\,,\\
\mathcal{R}_{10} &= \cos\vartheta\,, & \mathcal{S}_{10} &= 0\,,\\
\mathcal{R}_{11} &= \sin\vartheta\,\cos\lambda\,, & \mathcal{S}_{11} &= \sin\vartheta\,\sin\lambda\,,\\
\mathcal{R}_{20} &= \tfrac{3}{2}\cos^2\vartheta - \tfrac{1}{2}\,, & \mathcal{S}_{20} &= 0\,,\\
\mathcal{R}_{21} &= 3\sin\vartheta\,\cos\vartheta\,\cos\lambda\,, & \mathcal{S}_{21} &= 3\sin\vartheta\,\cos\vartheta\,\sin\lambda\,,\\
\mathcal{R}_{22} &= 3\sin^2\vartheta\,\cos 2\lambda\,, & \mathcal{S}_{22} &= 3\sin^2\vartheta\,\sin 2\lambda\,.
\end{aligned} \tag{2–81}$$

The corresponding solid harmonics $r^n\,\mathcal{R}_{nm}$ and $r^n\,\mathcal{S}_{nm}$ are simply homogeneous polynomials in x, y, z. For instance,

$$r^2\mathcal{S}_{22} = 6r^2\sin^2\vartheta\,\sin\lambda\,\cos\lambda = 6(r\sin\vartheta\,\cos\lambda)(r\sin\vartheta\,\sin\lambda) = 6xy\,. \tag{2–82}$$

In this way, we find

$$\begin{aligned}
\mathcal{R}_{00} &= 1\,, & \mathcal{S}_{00} &= 0\,,\\
r\,\mathcal{R}_{10} &= z\,, & r\,\mathcal{S}_{10} &= 0\,,\\
r\,\mathcal{R}_{11} &= x\,, & r\,\mathcal{S}_{11} &= y\,,\\
r^2\mathcal{R}_{20} &= -\tfrac{1}{2}\,x^2 - \tfrac{1}{2}\,y^2 + z^2, & r^2\mathcal{S}_{20} &= 0\,,\\
r^2\mathcal{R}_{21} &= 3x\,z\,, & r^2\mathcal{S}_{21} &= 3y\,z\,,\\
r^2\mathcal{R}_{22} &= 3x^2 - 3y^2\,, & r^2\mathcal{S}_{22} &= 6x\,y\,.
\end{aligned} \tag{2–83}$$

Substituting these functions into the expression (2–77) for the coefficients A_{nm} and B_{nm} yields for the zero-degree term

$$A_{00} = G \iiint_{\text{earth}} dM = GM\,; \tag{2–84}$$

that is, the product of the mass of the earth times the gravitational constant. For the first-degree coefficients, we get

$$A_{10} = G \iiint_{\text{earth}} z'\, dM\,, \quad A_{11} = G \iiint_{\text{earth}} x'\, dM\,, \quad B_{11} = G \iiint_{\text{earth}} y'\, dM\,; \tag{2–85}$$

and for the second-degree coefficients

$$\begin{aligned} A_{20} &= \frac{1}{2} G \iiint_{\text{earth}} (-x'^2 - y'^2 + 2z'^2)\, dM\,, \\ A_{21} &= G \iiint_{\text{earth}} x'\, z'\, dM\,, \qquad B_{21} = G \iiint_{\text{earth}} y'\, z'\, dM\,, \\ A_{22} &= \frac{1}{4} G \iiint_{\text{earth}} (x'^2 - y'^2)\, dM\,, \qquad B_{22} = \frac{1}{2} G \iiint_{\text{earth}} x'\, y'\, dM\,. \end{aligned} \tag{2–86}$$

It is known from mechanics that

$$x_c = \frac{1}{M} \iiint x'\, dM\,, \quad y_c = \frac{1}{M} \iiint y'\, dM\,, \quad z_c = \frac{1}{M} \iiint z'\, dM \tag{2–87}$$

are the rectangular coordinates of the center of gravity (center of mass, geocenter). If the origin of the coordinate system coincides with the center of gravity, then these coordinates and, hence, the integrals (2–85) are zero. *If the origin $r = 0$ is the center of gravity of the earth, then there will be no first-degree terms in the spherical-harmonic expansion of the potential V.* Therefore, this is true for our geocentric coordinate system.

The integrals

$$\iiint x'y'\, dM\,, \quad \iiint y'z'\, dM\,, \quad \iiint z'x'\, dM \tag{2–88}$$

are the *products of inertia*. They are zero if the coordinate axes coincide with the principal axes of inertia. If the z-axis is identical with the mean rotational axis of the earth, which coincides with the axis of maximum inertia, at least the second and third of these products of inertia must vanish. Hence, A_{21} and B_{21} will be zero, but not so B_{22}, which is proportional to the first product of inertia; B_{22} would vanish only if the earth had complete rotational symmetry or if a principal axis of inertia happened to coincide with the Greenwich meridian.

The five harmonics $A_{10}\,\mathcal{R}_{10}$, $A_{11}\,\mathcal{R}_{11}$, $B_{11}\,\mathcal{S}_{11}$, $A_{21}\,\mathcal{R}_{21}$, and $B_{21}\,\mathcal{S}_{21}$ – all first-degree harmonics and those of degree 2 and order 1 – which must, thus, vanish in any spherical-harmonic expansion of the earth's potential, are called *forbidden* or *inadmissible harmonics.*

Introducing the *moments of inertia* with respect to the x-, y-, z-axes by the definitions

$$\begin{aligned} A &= \iiint (y'^2 + z'^2)\, dM\,, \\ B &= \iiint (z'^2 + x'^2)\, dM\,, \\ C &= \iiint (x'^2 + y'^2)\, dM\,, \end{aligned} \tag{2–89}$$

and denoting the xy-product of inertia, which cannot be said to vanish, by

$$D = \iiint x'y'\, dM\,, \tag{2–90}$$

we finally have

$$\begin{aligned} A_{00} &= GM\,, \\ A_{10} &= A_{11} = B_{11} = 0\,, \\ A_{20} &= G\left[(A+B)/2 - C\right], \\ A_{21} &= B_{21} = 0\,, \\ A_{22} &= \tfrac{1}{4}\,G\,(B-A)\,, \\ B_{22} &= \tfrac{1}{2}\,G\,D\,. \end{aligned} \tag{2–91}$$

Now let the x- and y-axes actually coincide with the corresponding principal axes of inertia of the earth. This is only theoretically possible, since the principal axes of inertia of the earth are only inaccurately known. Then $B_{22} = 0$; taking into account (2–78) and (2–79), we may write explicitly

$$\begin{aligned} V = \frac{GM}{r} + \frac{G}{r^3}\bigg\{ &\frac{1}{2}\big[C - (A+B)/2\big](1 - 3\cos^2\vartheta) + \\ &\frac{3}{4}\,(B-A)\sin^2\vartheta\,\cos 2\lambda \bigg\} + O(1/r^4)\,. \end{aligned} \tag{2–92}$$

In rectangular coordinates this assumes the symmetrical form

$$V = \frac{GM}{r} + \frac{G}{2r^5}\Big[(B + C - 2A)\,x^2 + (C + A - 2B)\,y^2 + (A + B - 2C)\,z^2\Big] + O(1/r^4)\,, \tag{2–93}$$

which is obtained by taking into account the relations (1–26) between rectangular and spherical coordinates.

Terms of order higher than $1/r^3$ may be neglected for larger distances (say, for the distance to the moon), so that (2–92) or (2–93), omitting the higher-order terms $0(1/r^4)$, are sufficient for many astronomical purposes, cf. Moritz and Mueller (1987). Note that the notation $0(1/r^4)$ means terms of the order of $1/r^4$. For planetary distances even the first term,

$$V = \frac{GM}{r}\,, \tag{2–94}$$

is generally sufficient; it represents the potential of a point mass. Thus, for very large distances, every body acts like a point mass.

Using the form (2–78) of the spherical-harmonic expansion of V, then the coefficients of lower degree are obtained from (2–79) and (2–91). We find

$$\begin{aligned}
C_{10} &= C_{11} = S_{11} = 0\,,\\
C_{20} &= -\frac{C - (A + B)/2}{M\,a^2}\,,\\
C_{21} &= S_{21} = 0\,,\\
C_{22} &= \frac{B - A}{4M\,a^2}\,,\\
S_{22} &= \frac{D}{2M\,a^2}\,.
\end{aligned} \tag{2–95}$$

The first of these formulas shows that the summation in (2–78) actually begins with $n = 2$; the others relate the coefficients of second degree to the mass and the moments and products of inertia of the earth.

2.7 The gravity field of the level ellipsoid

As a first approximation, the earth is a sphere; as a second approximation, it may be considered an ellipsoid of revolution. Although the earth is not an exact ellipsoid, the gravity field of an ellipsoid is of fundamental practical importance because it is easy to handle mathematically and the deviations of the actual gravity field from the ellipsoidal "normal" field are so small that they can be considered linear. This splitting of the earth's gravity field into a "normal" and a remaining small "disturbing" field considerably simplifies the problem of its determination; the problem could hardly be solved otherwise.

Therefore, we assume that the normal figure of the earth is a level ellipsoid, that is, an ellipsoid of revolution which is an equipotential surface of a normal gravity field. This assumption is necessary because the ellipsoid is to be the normal form of the geoid, which is an equipotential surface of the actual gravity field. Denoting the potential of the normal gravity field by

$$U = U(x,\, y,\, z)\,, \tag{2–96}$$

we see that the level ellipsoid, being a surface $U =$ constant, exactly corresponds to the geoid, which is defined as a surface $W =$ constant.

The basic point here is that by postulating that the given ellipsoid be an equipotential surface of the normal gravity field, and by prescribing the total mass M, we completely and uniquely determine the normal potential U. The detailed density distribution inside the ellipsoid, which produces the potential U, is quite uninteresting and need not be known at all. In fact, we do not know of any "reasonable" mass distribution for the level ellipsoid (Moritz 1990: Chap. 5). Pizzetti (1894) unsuccessfully used a homogeneous density distribution combined with a surface layer of negative density, which is quite "unnatural".

This determination is possible by Dirichlet's principle (Sect. 1.12): The gravitational potential outside a surface S is completely determined by knowing the geometric shape of S and the value of the potential on S. Originally it was shown only for the gravitational potential V, but it can be applied to the gravity potential

$$U = V + \tfrac{1}{2}\,\omega^2(x^2 + y^2) \tag{2–97}$$

as well if the angular velocity ω is given. The proof follows that in Sect. 1.12, with obvious modifications. Hence, the normal potential function $U(x, y, z)$ is completely determined by

1. the shape of the ellipsoid of revolution, that is, its semiaxes a and b,
2. the total mass M, and
3. the angular velocity ω.

The calculation will now be carried out in detail. The given ellipsoid S_0,

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1\,, \tag{2–98}$$

is by definition an equipotential surface

$$U(x,\, y,\, z) = U_0\,. \tag{2–99}$$

It is now convenient to introduce the ellipsoidal-harmonic coordinates u, β, λ of Sect. 1.15. The ellipsoid S_0 is taken as the reference ellipsoid $u = b$.

Since $V(u,\beta)$, the gravitational part of the normal potential U, will be harmonic outside the ellipsoid S_0, we use the second equation of the series (1–174). The field V has rotational symmetry and, hence, does not depend on the longitude λ. Therefore, all nonzonal terms, which depend on λ, must be zero, and there remains

$$V(u,\beta) = \sum_{n=0}^{\infty} \frac{Q_n\left(i\,\dfrac{u}{E}\right)}{Q_n\left(i\,\dfrac{b}{E}\right)}\, A_n P_n(\sin\beta)\,, \tag{2–100}$$

where

$$E = \sqrt{a^2 - b^2} \tag{2–101}$$

is the linear eccentricity. The centrifugal potential $\Phi(u,\beta)$ is given by

$$\Phi(u,\, \beta) = \tfrac{1}{2}\,\omega^2(u^2 + E^2)\,\cos^2\!\beta\,. \tag{2–102}$$

Therefore, the total normal gravity potential may be written

$$U(u,\, \beta) = \sum_{n=0}^{\infty} \frac{Q_n\left(i\,\dfrac{u}{E}\right)}{Q_n\left(i\,\dfrac{b}{E}\right)}\, A_n P_n(\sin\beta) + \tfrac{1}{2}\,\omega^2(u^2 + E^2)\,\cos^2\!\beta\,. \tag{2–103}$$

On the ellipsoid S_0 we have $u = b$ and $U = U_0$. Hence,

$$\sum_{n=0}^{\infty} A_n P_n(\sin\beta) + \tfrac{1}{2}\,\omega^2(u^2 + E^2)\,\cos^2\!\beta = U_0\,. \tag{2–104}$$

This equation applies for all points of S_0, that is, for all values of β. Since

$$b^2 + E^2 = a^2 \tag{2–105}$$

and

$$\cos^2\!\beta = \tfrac{2}{3}\left[1 - P_2(\sin\beta)\right], \tag{2–106}$$

we have

$$\sum_{n=0}^{\infty} A_n P_n(\sin\beta) + \tfrac{1}{3}\,\omega^2 a^2 - \tfrac{1}{3}\,\omega^2 a^2 P_2(\sin\beta) - U_0 = 0 \tag{2–107}$$

or

$$\left(A_0 + \tfrac{1}{3}\,\omega^2 a^2 - U_0\right) P_0(\sin\beta) + A_1 P_1(\sin\beta)$$
$$+ \left(A_2 - \tfrac{1}{3}\,\omega^2 a^2\right) P_2(\sin\beta) + \sum_{n=3}^{\infty} A_n P_n(\sin\beta) = 0\,. \qquad (2\text{–}108)$$

This equation applies for all values of β only if the coefficient of every $P_n(\sin\beta)$ is zero. Thus, we get

$$A_0 = U_0 - \tfrac{1}{3}\,\omega^2 a^2\,, \qquad A_1 = 0\,,$$
$$A_2 = \tfrac{1}{3}\,\omega^2 a^2\,, \qquad A_3 = A_4 = \ldots = 0\,. \qquad (2\text{–}109)$$

Substituting these relations into (2–100) gives

$$V(u,\beta) = \left(U_0 - \tfrac{1}{3}\,\omega^2 a^2\right) \frac{Q_0\left(i\,\dfrac{u}{E}\right)}{Q_0\left(i\,\dfrac{b}{E}\right)} + \tfrac{1}{3}\,\omega^2 a^2\,\frac{Q_2\left(i\,\dfrac{u}{E}\right)}{Q_2\left(i\,\dfrac{b}{E}\right)}\,P_2(\sin\beta)\,. \quad (2\text{–}110)$$

This formula is basically the solution of Dirichlet's problem for the level ellipsoid, but we can give it more convenient forms. It is a closed formula!

First, we determine the Legendre functions of the second kind, Q_0 and Q_2. As

$$\coth^{-1}(i\,x) = \frac{1}{i}\,\cot^{-1}x = -i\,\tan^{-1}\frac{1}{x}\,, \qquad (2\text{–}111)$$

we find by (1–80) with $z = i\,u/E$:

$$Q_0\left(i\,\frac{u}{E}\right) = -i\tan^{-1}\frac{E}{u}\,,$$
$$Q_2\left(i\,\frac{u}{E}\right) = \frac{i}{2}\left[\left(1 + 3\,\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - 3\,\frac{u}{E}\right]. \qquad (2\text{–}112)$$

By introducing in (2–112) the abbreviations

$$q = \frac{1}{2}\left[\left(1 + 3\,\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - 3\,\frac{u}{E}\right],$$
$$q_0 = \frac{1}{2}\left[\left(1 + 3\,\frac{b^2}{E^2}\right)\tan^{-1}\frac{E}{b} - 3\,\frac{b}{E}\right] \qquad (2\text{–}113)$$

and substituting them in equation (2–110), we obtain

$$V(u,\beta) = \left(U_0 - \tfrac{1}{3}\,\omega^2 a^2\right) \frac{\tan^{-1}\dfrac{E}{u}}{\tan^{-1}\dfrac{E}{b}} + \tfrac{1}{3}\,\omega^2 a^2\,\tfrac{q}{q_0}\,P_2(\sin\beta)\,. \qquad (2\text{–}114)$$

Now we can express U_0 in terms of the mass M. For large values of u, we have

$$\tan^{-1}\frac{E}{u} = \frac{E}{u} + O(1/u^3)\,. \tag{2–115}$$

From the expressions (1–26) for spherical coordinates and from equations (1–151) for ellipsoidal-harmonic coordinates, we find

$$x^2 + y^2 + z^2 = r^2 = u^2 + E^2 \cos^2\beta\,, \tag{2–116}$$

so that for large values of r we have

$$\frac{1}{u} = \frac{1}{r} + O(1/r^3) \tag{2–117}$$

and

$$\tan^{-1}\frac{E}{u} = \frac{E}{r} + O(1/r^3)\,, \tag{2–118}$$

where $O(x)$ means "small of order x", i.e., small of order $1/r^3$ in our case. For very large distances r, the first term in (2–114) is dominant, so that asymptotically

$$V = \left(U_0 - \tfrac{1}{3}\,\omega^2 a^2\right) \frac{E}{\tan^{-1}(E/b)}\,\frac{1}{r} + O(1/r^3)\,. \tag{2–119}$$

We know from Sect. 2.6 that

$$V = \frac{GM}{r} + O(1/r^3)\,. \tag{2–120}$$

Substituting this expression for V into the left-hand side of (2–119) yields

$$\frac{GM}{r} = \left(U_0 - \tfrac{1}{3}\,\omega^2 a^2\right) \frac{E}{\tan^{-1}(E/b)}\,\frac{1}{r} + O(1/r^3)\,. \tag{2–121}$$

Now multiply this equation by r and let then $r \to 0$. The result is (rigorously!)

$$GM = \left(U_0 - \tfrac{1}{3}\,\omega^2 a^2\right) \frac{E}{\tan^{-1}(E/b)}\,, \tag{2–122}$$

which may be rearranged to

$$U_0 = \frac{GM}{E}\,\tan^{-1}\frac{E}{b} + \tfrac{1}{3}\,\omega^2 a^2\,. \tag{2–123}$$

This is the desired relation between mass M and potential U_0.

Substituting the result for U_0 obtained in (2–123) into (2–114), simplifies the expression for V to

$$V = \frac{GM}{E}\,\tan^{-1}\frac{E}{u} + \tfrac{1}{3}\,\omega^2 a^2\,\frac{q}{q_0}\,P_2(\sin\beta)\,. \tag{2–124}$$

Expressing P_2 as

$$P_2(\sin\beta) = \tfrac{3}{2}\,\sin^2\beta - \tfrac{1}{2} \tag{2–125}$$

and, finally, adding the centrifugal potential $\Phi = \omega^2(u^2 + E^2)\,\cos^2\beta/2$ from (2–102), the normal gravity potential U results as

$$U(u,\,\beta) = \frac{GM}{E}\,\tan^{-1}\frac{E}{u} + \tfrac{1}{2}\,\omega^2 a^2 \frac{q}{q_0}\left(\sin^2\beta - \tfrac{1}{3}\right) + \tfrac{1}{2}\,\omega^2(u^2 + E^2)\,\cos^2\beta\,. \tag{2–126}$$

The only constants that occur in this formula are a, b, GM, and ω. This is in complete agreement with Dirichlet's theorem.

2.8 Normal gravity

Referring to the line element in ellipsoidal-harmonic coordinates according to (1–155), replacing ϑ by its complement $90° - \beta$, we get

$$ds^2 = w^2\,du^2 + w^2(u^2 + E^2)\,d\beta^2 + (u^2 + E^2)\,\cos^2\beta\,d\lambda^2\,, \tag{2–127}$$

where

$$w = \sqrt{\frac{u^2 + E^2\sin^2\beta}{u^2 + E^2}} \tag{2–128}$$

has been introduced. Thus, along the coordinate lines we have

$$\begin{aligned}
&u = \text{variable},\ \beta = \text{constant},\ \lambda = \text{constant}, \quad ds_u = w\,du\,,\\
&\beta = \text{variable},\ u = \text{constant},\ \lambda = \text{constant}, \quad ds_\beta = w\,\sqrt{u^2 + E^2}\,d\beta\,,\\
&\lambda = \text{variable},\ u = \text{constant},\ \beta = \text{constant}, \quad ds_\lambda = \sqrt{u^2 + E^2}\,\cos\beta\,d\lambda\,.
\end{aligned} \tag{2–129}$$

The components of the normal gravity vector

$$\boldsymbol{\gamma} = \operatorname{grad} U \tag{2–130}$$

along these coordinate lines are accordingly given by

$$\begin{aligned}
\gamma_u &= \frac{\partial U}{\partial s_u} = \frac{1}{w}\,\frac{\partial U}{\partial u}\,,\\
\gamma_\beta &= \frac{\partial U}{\partial s_\beta} = \frac{1}{w\sqrt{u^2 + E^2}}\,\frac{\partial U}{\partial \beta}\,,\\
\gamma_\lambda &= \frac{\partial U}{\partial s_\lambda} = \frac{1}{\sqrt{u^2 + E^2}\,\cos\beta}\,\frac{\partial U}{\partial \lambda} = 0\,.
\end{aligned} \tag{2–131}$$

The component γ_λ is zero because U does not contain λ. This is also evident from the rotational symmetry.

Performing the partial differentiations, we find

$$
\begin{aligned}
-w\,\gamma_u &= \frac{GM}{u^2+E^2} + \frac{\omega^2 a^2 E}{u^2+E^2}\,\frac{q'}{q_0}\left(\tfrac{1}{2}\sin^2\beta - \tfrac{1}{6}\right) - \omega^2 u\cos^2\beta\,,\\
-w\,\gamma_\beta &= \left(-\frac{\omega^2 a^2}{\sqrt{u^2+E^2}}\,\frac{q}{q_0} + \omega^2\sqrt{u^2+E^2}\right)\sin\beta\,\cos\beta\,,
\end{aligned}
\tag{2–132}
$$

where we have set

$$
q' = -\frac{u^2+E^2}{E}\,\frac{dq}{du} = 3\left(1+\frac{u^2}{E^2}\right)\left(1-\frac{u}{E}\tan^{-1}\frac{E}{u}\right) - 1\,. \tag{2–133}
$$

Note that q' does not mean dq/du; this notation has been borrowed from Hirvonen (1960), where q' is the derivative with respect to another independent variable which we are not using here.

For the level ellipsoid S_0 itself, we have $u = b$ and get

$$
\gamma_{\beta,0} = 0\,. \tag{2–134}
$$

(Note that we will often mark quantities referred to S_0 by the subscript 0.) This is also evident because on S_0 the gravity vector is normal to the level surface S_0. Hence, in addition to the λ-component, the β-component is also zero on the reference ellipsoid $u = b$. Note that the other coordinate ellipsoids $u =$ constant are not equipotential surfaces $U =$ constant, so that the β-component will not in general be zero.

Thus, the total gravity on the ellipsoid S_0, which we simply denote by γ, is given by

$$
\begin{aligned}
\gamma = |\gamma_{u,0}| = {} & \frac{GM}{a\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}}\,\cdot\\
& \cdot\left[1 + \frac{\omega^2 a^2 E}{GM}\,\frac{q_0'}{q_0}\left(\tfrac{1}{2}\sin^2\beta - \tfrac{1}{6}\right) - \frac{\omega^2 a^2 b}{GM}\cos^2\beta\right],
\end{aligned}
\tag{2–135}
$$

since on S_0 we get the relations

$$
\begin{aligned}
\sqrt{u^2+E^2} &= \sqrt{b^2+E^2} = a\,,\\
w_0 &= \frac{1}{a}\sqrt{b^2+E^2\sin^2\beta} = \frac{1}{a}\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}\,.
\end{aligned}
\tag{2–136}
$$

Now we introduce the abbreviation

$$
m = \frac{\omega^2 a^2 b}{GM} \tag{2–137}
$$

and the second eccentricity

$$e' = \frac{E}{b} = \frac{\sqrt{a^2 - b^2}}{b} \, . \tag{2–138}$$

The prime on e does not denote differentiation, but merely distinguishes the second eccentricity from the first eccentricity which is defined as $e = E/a$.

Removing the constant terms by noting that

$$1 = \cos^2\beta + \sin^2\beta \, , \tag{2–139}$$

we obtain

$$\begin{aligned} \gamma = & \frac{GM}{a\sqrt{a^2 \sin^2\beta + b^2 \cos^2\beta}} \cdot \\ & \cdot \left[\left(1 + \frac{m}{3} \frac{e' q_0'}{q_0} \right) \sin^2\beta + \left(1 - m - \frac{m}{6} \frac{e' q_0'}{q_0} \right) \cos^2\beta \right] . \end{aligned} \tag{2–140}$$

At the equator ($\beta = 0$), we find

$$\gamma_a = \frac{GM}{a\,b} \left(1 - m - \frac{m}{6} \frac{e' q_0'}{q_0} \right) ; \tag{2–141}$$

at the poles ($\beta = \pm 90°$), normal gravity is given by

$$\gamma_b = \frac{GM}{a^2} \left(1 + \frac{m}{3} \frac{e' q_0'}{q_0} \right) . \tag{2–142}$$

Normal gravity at the equator, γ_a, and normal gravity at the pole, γ_b, satisfy the relation

$$\frac{a - b}{a} + \frac{\gamma_b - \gamma_a}{\gamma_a} = \frac{\omega^2 b}{\gamma_a} \left(1 + \frac{e' q_0'}{2 q_0} \right) , \tag{2–143}$$

which should be verified by substitution. This is the rigorous form of an important approximate formula published by Clairaut in 1738. It is, therefore, called Clairaut's theorem. Its significance will become clear in Sect. 2.10.

By comparing expression (2–141) for γ_a and expression (2–142) for γ_b with the quantities within parentheses in formula (2–140), we see that γ can be written in the symmetrical form

$$\gamma = \frac{a\,\gamma_b \sin^2\beta + b\,\gamma_a \cos^2\beta}{\sqrt{a^2 \sin^2\beta + b^2 \cos^2\beta}} \, . \tag{2–144}$$

We finally introduce the ellipsoidal latitude on the ellipsoid, φ, which is the angle between the normal to the ellipsoid and the equatorial plane (Fig. 2.11). Using the formula from ellipsoidal geometry,

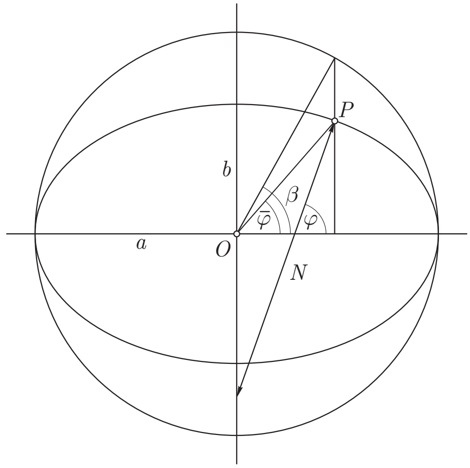

Fig. 2.11. Ellipsoidal latitude φ, geocentric latitude $\bar{\varphi}$, reduced (ellipsoidal-harmonic) latitude β for a point P on the ellipsoid

$$\tan\beta = \frac{b}{a}\tan\varphi\,, \tag{2–145}$$

we obtain

$$\gamma = \frac{a\,\gamma_a\cos^2\!\varphi + b\,\gamma_b\sin^2\!\varphi}{\sqrt{a^2\cos^2\!\varphi + b^2\sin^2\!\varphi}}\,. \tag{2–146}$$

The computation is left as an exercise for the reader. This rigorous formula for normal gravity on the ellipsoid is due to Somigliana from 1929.

We close this section with a short remark on the vertical gradient of gravity at the reference ellipsoid, $\partial\gamma/\partial s_u = \partial\gamma/\partial h$. Bruns' formula (2–40), applied to the normal gravity field with the corresponding ellipsoidal height h and with $\varrho = 0$, yields

$$\frac{\partial\gamma}{\partial h} = -2\gamma\,J - 2\omega^2\,. \tag{2–147}$$

The mean curvature of the ellipsoid is given by

$$J = \frac{1}{2}\left(\frac{1}{M} + \frac{1}{N}\right), \tag{2–148}$$

where M and N are the principal radii of curvature: M is the radius in the direction of the meridian, and N is the normal radius of curvature, taken in the direction of the prime vertical. From ellipsoidal geometry, we use the formulas

$$M = \frac{c}{(1+e'^2\cos^2\!\varphi)^{3/2}}\,, \qquad N = \frac{c}{(1+e'^2\cos^2\!\varphi)^{1/2}}\,, \tag{2–149}$$

where

$$c = \frac{a^2}{b} \tag{2–150}$$

is the radius of curvature at the pole. The normal radius of curvature, N, admits a simple geometrical interpretation (Fig. 2.11). It is, therefore, also known as the "normal terminated by the minor axis" (Bomford 1962: p. 497).

2.9 Expansion of the normal potential in spherical harmonics

We have found the gravitational potential of the normal figure of the earth in terms of ellipsoidal harmonics in (2–124) as

$$V = \frac{GM}{E} \tan^{-1}\frac{E}{u} + \frac{1}{3}\,\omega^2 a^2\,\frac{q}{q_0}\,P_2(\sin\beta)\,. \tag{2–151}$$

Now we wish to express this equation in terms of spherical coordinates r, ϑ, λ.

We first establish a relation between ellipsoidal-harmonic and spherical coordinates. By comparing the rectangular coordinates in these two systems according to Eqs. (1–26) and (1–151), we get

$$\begin{aligned} r\,\sin\vartheta\,\cos\lambda &= \sqrt{u^2+E^2}\,\cos\beta\,\cos\lambda\,,\\ r\,\sin\vartheta\,\sin\lambda &= \sqrt{u^2+E^2}\,\cos\beta\,\sin\lambda\,,\\ r\,\cos\vartheta &= u\,\sin\beta\,. \end{aligned} \tag{2–152}$$

The longitude λ is the same in both systems. We easily find from these equations

$$\begin{aligned} \cot\vartheta &= \frac{u}{\sqrt{u^2+E^2}}\,\tan\beta\,,\\ r &= \sqrt{u^2+E^2\cos^2\!\beta}\,. \end{aligned} \tag{2–153}$$

The direct transformation of (2–151) by expressing u and β in terms of r and ϑ by means of equations (2–153) is extremely laborious. However, the problem can be solved easily in an indirect way.

We expand $\tan^{-1}(E/u)$ into the well-known power series

$$\tan^{-1}\frac{E}{u} = \frac{E}{u} - \frac{1}{3}\left(\frac{E}{u}\right)^3 + \frac{1}{5}\left(\frac{E}{u}\right)^5 - \ldots\,. \tag{2–154}$$

The substitution of this series into the first equation of formula (2–113), i.e.,

$$q = \frac{1}{2}\left[\left(1+3\,\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - 3\,\frac{u}{E}\right], \tag{2–155}$$

leads, after simple manipulations, to

$$q = 2\left[\frac{1}{3\cdot 5}\left(\frac{E}{u}\right)^3 - \frac{2}{5\cdot 7}\left(\frac{E}{u}\right)^5 + \frac{3}{7\cdot 9}\left(\frac{E}{u}\right)^7 - \dots\right]. \tag{2–156}$$

More concisely, we have

$$\begin{aligned} \tan^{-1}\frac{E}{u} &= \frac{E}{u} + \sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{E}{u}\right)^{2n+1}, \\ q &= -\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1}. \end{aligned} \tag{2–157}$$

By inserting these relations into (2–151) we obtain

$$\begin{aligned} V = \frac{GM}{u} &+ \frac{GM}{E}\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{E}{u}\right)^{2n+1} \\ &- \frac{\omega^2 a^2}{3q_0}\sum_{n=1}^{\infty}(-1)^n \frac{2n}{(2n+1)(2n+3)}\left(\frac{E}{u}\right)^{2n+1} P_2(\sin\beta). \end{aligned} \tag{2–158}$$

Introducing m, defined by (2–137), and the second eccentricity $e' = E/b$, we find

$$\begin{aligned} V = \frac{GM}{u} &+ \sum_{n=1}^{\infty}(-1)^n \frac{GM}{(2n+1)\,E}\left(\frac{E}{u}\right)^{2n+1} \cdot \\ &\cdot \left[1 - \frac{m\,e'}{3q_0}\frac{2n}{2n+3}P_2(\sin\beta)\right]. \end{aligned} \tag{2–159}$$

We expand the potential V into a series of spherical harmonics. Because of the rotational symmetry, there will be only zonal terms, and because of the symmetry with respect to the equatorial plane, there will be only even zonal harmonics. The zonal harmonics of odd degree change sign for negative latitudes and must, therefore, be absent. Accordingly, the series has the form

$$V = \frac{GM}{r} + A_2\frac{P_2(\cos\vartheta)}{r^3} + A_4\frac{P_4(\cos\vartheta)}{r^5} + \cdots. \tag{2–160}$$

We next have to determine the coefficients A_2, A_4, ... For this purpose, we consider a point on the axis of rotation, outside the ellipsoid. For this point, we have $\beta = 90°$, $\vartheta = 0°$, and, by (2–153), $u = r$. Then (2–159) becomes

$$V = \frac{GM}{r} + \sum_{n=1}^{\infty}(-1)^n \frac{GM\,E^{2n}}{2n+1}\left(1 - \frac{2n}{2n+3}\,\frac{m\,e'}{3q_0}\right)\frac{1}{r^{2n+1}}, \tag{2–161}$$

and (2–160) takes the form

$$V = \frac{GM}{r} + \frac{A_2}{r^3} + \frac{A_4}{r^5} + \cdots = \frac{GM}{r} + \sum_{n=1}^{\infty} A_{2n} \frac{1}{r^{2n+1}} . \qquad (2\text{–}162)$$

Here we have used the fact that for all values of n

$$P_n(1) = 1 \qquad (2\text{–}163)$$

(see also Fig. 1.4). Comparing the coefficients in both expressions for V, we find

$$A_{2n} = (-1)^n \frac{GM\,E^{2n}}{2n+1} \left(1 - \frac{2n}{2n+3}\,\frac{m\,e'}{3q_0}\right) . \qquad (2\text{–}164)$$

Equations (2–160) and (2–164) give the desired expression for the potential of the level ellipsoid as a series of spherical harmonics.

The second-degree coefficient A_2 is

$$A_2 = G\,(A - C) . \qquad (2\text{–}165)$$

This follows from (2–91) by using $A = B$ for reasons of symmetry. The C is the moment of inertia with respect to the axis of rotation, and A is the moment of inertia with respect to any axis in the equatorial plane. By letting $n = 1$ in (2–164), we obtain

$$A_2 = -\frac{1}{3}\,GM\,E^2 \left(1 - \frac{2}{15}\,\frac{m\,e'}{q_0}\right) . \qquad (2\text{–}166)$$

Comparing this with the preceding Eq. (2–165), we find

$$G\,(C - A) = \frac{1}{3}\,GM\,E^2 \left(1 - \frac{2}{15}\,\frac{m\,e'}{q_0}\right) . \qquad (2\text{–}167)$$

Thus, the difference between the principal moments of inertia is expressed in terms of "Stokes' constants" a, b, M, and ω.

It is possible to eliminate q_0 from Eqs. (2–164) and (2–167), obtaining

$$A_{2n} = (-1)^n \frac{3GM\,E^{2n}}{(2n+1)(2n+3)} \left(1 - n + 5n\,\frac{C - A}{M\,E^2}\right) . \qquad (2\text{–}168)$$

If we write the potential V in the form

$$\begin{aligned} V &= \frac{GM}{r}\left[1 + C_2\left(\frac{a}{r}\right)^2 P_2(\cos\vartheta) + C_4\left(\frac{a}{r}\right)^4 P_4(\cos\vartheta) + \cdots\right] \\ &= \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty} C_{2n}\left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\vartheta)\right] , \end{aligned} \qquad (2\text{–}169)$$

then the C_{2n} are given by

$$C_{2n} = -J_{2n} = (-1)^n \frac{3e^{2n}}{(2n+1)(2n+3)} \left(1 - n + 5n\,\frac{C-A}{M\,E^2}\right) . \tag{2–170}$$

Here we have introduced the first eccentricity $e = E/a$. For $n = 1$ this gives the important formula

$$C_{20} = -\frac{C-A}{M\,a^2} \tag{2–171}$$

or, equivalently,

$$J_2 = \frac{C-A}{M\,a^2}\,, \tag{2–172}$$

which is in agreement with the respective relation in (2–95) when taking into account the rotational symmetry causing $A = B$.

Finally, we note that on eliminating $q_0 = (1/i)\,Q_2(i(b/E))$ by using Eq. (2–167), and U_0 by using Eq. (2–122), we may write the expansion of V in ellipsoidal harmonics, Eq. (2–110), in the form

$$\begin{aligned} V(u,\,\beta) = {} & \frac{i}{E}\,GM\,Q_0\left(i\,\frac{u}{E}\right) \\ & + \frac{15i}{2E^3}\,G\left(C - A - \tfrac{1}{3}\,M\,E^2\right) Q_2\left(i\,\frac{u}{E}\right) P_2(\sin\beta)\,. \end{aligned} \tag{2–173}$$

This shows that the coefficients of the ellipsoidal harmonics of degrees zero and two are functions of the mass and of the difference between the two principal moments of inertia. The analogy to the corresponding spherical-harmonic coefficients (2–91) is obvious. *This is a closed formula, not a truncated series!*

2.10 Series expansions for the normal gravity field

Since the earth ellipsoid is very nearly a sphere, the quantities

$$\begin{aligned} E &= \sqrt{a^2 - b^2}\,, && \text{linear eccentricity,} \\ e &= \frac{E}{a}\,, && \text{first (numerical) eccentricity,} \\ e' &= \frac{E}{b}\,, && \text{second (numerical) eccentricity,} \\ f &= \frac{a-b}{a}\,, && \text{flattening,} \end{aligned} \tag{2–174}$$

and similar parameters that characterize the deviation from a sphere are small. Therefore, series expansions in terms of these or similar parameters will be convenient for numerical calculations.

Linear approximation

In order that the readers may find their way through the subsequent practical formulas, we first consider an approximation that is linear in the flattening f. Here we get particularly simple and symmetrical formulas which also exhibit plainly the structure of the higher-order expansions.

It is well known that the radius vector r of an ellipsoid is approximately given by

$$r = a\,(1 - f\sin^2\varphi)\,. \tag{2–175}$$

As we will see subsequently, normal gravity may, to the same approximation, be written

$$\gamma = \gamma_a\,(1 + f^*\sin^2\varphi)\,. \tag{2–176}$$

For $\varphi = \pm 90°$, at the poles, we have $r = b$ and $\gamma = \gamma_b$. Hence, we may write

$$b = a\,(1 - f)\,, \qquad \gamma_b = \gamma_a\,(1 + f^*)\,, \tag{2–177}$$

and solving for f and f^*, we obtain

$$\begin{aligned} f\; &= \frac{a-b}{a}\,,\\ f^* &= \frac{\gamma_b - \gamma_a}{\gamma_a}\,, \end{aligned} \tag{2–178}$$

so that f is the flattening defined by (2–174), and f^* is an analogous quantity which may be called *gravity flattening*.

To the same approximation, (2–143) becomes

$$f + f^* = \tfrac{5}{2}\,m\,, \tag{2–179}$$

where

$$m \doteq \frac{\omega^2 a}{\gamma_a} = \frac{\text{centrifugal force at equator}}{\text{gravity at equator}}\,. \tag{2–180}$$

This is *Clairaut's theorem* in its original form. It is one of the most striking formulas of physical geodesy: the (geometrical) flattening f in (2–178) can be derived from f^* and m, which are purely dynamical quantities obtained by gravity measurements; that is, *the flattening of the earth can be obtained from gravity measurements.*

Clairaut's formula is only a first approximation and must be improved, first by the inclusion of higher-order ellipsoidal terms in f, and secondly by taking into account the deviation of the earth's gravity field from the normal gravity field. But the principle remains the same.

Second-order expansion

We now expand the closed formulas of the two preceding sections into series in terms of the second numerical eccentricity e' and the flattening f, in general up to and including e'^4 or f^2. Terms of the order of e'^6 or f^3 and higher will usually be neglected.

We start from the series

$$\tan^{-1}\frac{E}{u} = \frac{E}{u} - \frac{1}{3}\left(\frac{E}{u}\right)^3 + \frac{1}{5}\left(\frac{E}{u}\right)^5 - \frac{1}{7}\left(\frac{E}{u}\right)^7 + \cdots\,,$$

$$q = 2\left[\frac{1}{3\cdot 5}\left(\frac{E}{u}\right)^3 - \frac{2}{5\cdot 7}\left(\frac{E}{u}\right)^5 + \frac{3}{7\cdot 9}\left(\frac{E}{u}\right)^7 - \cdots\right], \qquad (2\text{–}181)$$

$$q' = 6\left[\frac{1}{3\cdot 5}\left(\frac{E}{u}\right)^3 - \frac{1}{5\cdot 7}\left(\frac{E}{u}\right)^5 + \frac{1}{7\cdot 9}\left(\frac{E}{u}\right)^7 - \cdots\right].$$

The first two series have already been used in the preceding section in (2–154) and (2–156), respectively; the third is obtained by substituting the $\tan^{-1}$ series into the closed formula (2–133) for q'.

On the reference ellipsoid S_0, we have $u = b$ and

$$\frac{E}{u} = \frac{E}{b} = e'\,, \qquad (2\text{–}182)$$

so that

$$\tan^{-1}e' = e' - \tfrac{1}{3}\,e'^3 + \tfrac{1}{5}\,e'^5\ \cdots\,,$$

$$q_0 = \tfrac{2}{15}\,e'^3\left(1 - \tfrac{6}{7}\,e'^2\ \cdots\right),$$

$$q'_0 = \tfrac{2}{5}\,e'^2\left(1 - \tfrac{3}{7}\,e'^2\ \cdots\right), \qquad (2\text{–}183)$$

$$\frac{e'\,q'_0}{q_0} = 3\left(1 + \tfrac{3}{7}\,e'^2\ \cdots\right).$$

We also need the series

$$b = \frac{a}{\sqrt{1+e'^2}} = a\left(1 - \tfrac{1}{2}\,e'^2 + \tfrac{3}{8}\,e'^4\ \cdots\right). \qquad (2\text{–}184)$$

Potential and gravity

By substituting these expressions into the closed formulas (2–123), (2–141), (2–142), and (2–143), we obtain, up to and including the order e'^4, the following relations.

Potential:

$$U_0 = \frac{GM}{b}\left(1 - \tfrac{1}{3}\,e'^2 + \tfrac{1}{5}\,e'^4\right) + \tfrac{1}{3}\,\omega^2 a^2\,. \tag{2–185}$$

Gravity at the equator and the pole:

$$\begin{aligned}\gamma_a &= \frac{GM}{a\,b}\left(1 - \tfrac{3}{2}\,m - \tfrac{3}{14}\,e'^2 m\right),\\ \gamma_b &= \frac{GM}{a^2}\left(1 + m + \tfrac{3}{7}\,e'^2 m\right).\end{aligned} \tag{2–186}$$

Clairaut's theorem:

$$f + f^* = \frac{5}{2}\,\frac{\omega^2 b}{\gamma_a}\left(1 + \frac{9}{35}\,e'^2\right). \tag{2–187}$$

The ratio $\omega^2 a/\gamma_a$ may be expressed as

$$\frac{\omega^2 a}{\gamma_a} = m + \tfrac{3}{2}\,m^2\,, \tag{2–188}$$

which is a more accurate version of (2–180).

From the first equation of (2–186), we find

$$GM = a\,b\,\gamma_a\left(1 + \tfrac{3}{2}\,m + \tfrac{3}{14}\,e'^2 m + \tfrac{9}{4}\,m^2\right), \tag{2–189}$$

which gives the mass in terms of equatorial gravity. Using this equation, we can express GM in Eq. (2–185) in terms of γ_a, obtaining

$$U_0 = a\,\gamma_a\left(1 - \tfrac{1}{3}\,e'^2 + \tfrac{11}{6}\,m + \tfrac{1}{5}\,e'^4 - \tfrac{2}{7}\,e'^2 m + \tfrac{11}{4}\,m^2\right). \tag{2–190}$$

Here we have eliminated $\omega^2 a$ by replacing it with $GM\,m/b$.

Now we can turn to Eq. (2–146) for normal gravity. A simple manipulation yields

$$\gamma = \gamma_a\,\frac{1 + \frac{b\,\gamma_b - a\,\gamma_a}{a\,\gamma_b}\,\sin^2\varphi}{\sqrt{1 - \frac{a^2 - b^2}{a^2}\,\sin^2\varphi}}\,. \tag{2–191}$$

The denominator is expanded into a binomial series:

$$\frac{1}{\sqrt{1-x}} = 1 + \tfrac{1}{2}\,x + \tfrac{3}{8}\,x^2 + \cdots\,. \tag{2–192}$$

Then the abbreviated series

$$\frac{a^2-b^2}{a^2} = \frac{e'^2}{1+e'^2} = e'^2 - e'^4 \,,$$
$$\frac{b\,\gamma_b - a\,\gamma_a}{a\,\gamma_a} = -e'^2 + \tfrac{5}{2}\,m^2 + e'^4 - \tfrac{13}{7}\,e'^2 m + \tfrac{15}{4}\,m^2 \tag{2–193}$$

are introduced and we obtain, upon substitution,

$$\gamma = \gamma_a \Big[1 + \big(-\tfrac{1}{2}\,e'^2 + \tfrac{5}{2}\,m + \tfrac{1}{2}\,e'4 - \tfrac{13}{7}\,e'^2 m + \tfrac{15}{4}\,m^2 \big) \sin^2\varphi$$
$$+ \big(-\tfrac{1}{8}\,e'^4 + \tfrac{5}{4}\,e'^2 m \big) \sin^4\varphi \Big] \,. \tag{2–194}$$

We may also express these quantities in terms of the flattening f by substituting the equation

$$e'^2 = \frac{1}{(1-f)^2} - 1 = 2f + 3f^2 + \cdots \,. \tag{2–195}$$

The flattening f is most commonly used; it offers a slight advantage over the second eccentricity e' in that it is of the same order of magnitude as m: it is not immediately apparent that m^2, $e'^2 m$, and e'^4 are quantities of the same order of magnitude. We obtain

$$GM = a\,b\,\gamma_a \big(1 + \tfrac{3}{2}\,m + \tfrac{3}{7}\,f\,m + + \tfrac{9}{4}\,m^2 \big) \,, \tag{2–196}$$

$$U_0 = a\,\gamma_a \big(1 - \tfrac{2}{3}\,f + \tfrac{11}{6}\,m - \tfrac{1}{5}\,f^2 - \tfrac{4}{7}\,f\,m + \tfrac{11}{4}\,m^2 \big) \,, \tag{2–197}$$

$$\gamma = \gamma_a \Big[1 + \big(-f + \tfrac{5}{2}\,m + \tfrac{1}{2}\,f^2 - \tfrac{26}{7}\,f\,m + \tfrac{15}{4}\,m^2 \big) \sin^2\varphi$$
$$+ \big(-\tfrac{1}{2}\,f^2 + \tfrac{5}{2}\,f\,m \big) \sin^4\varphi \Big]. \tag{2–198}$$

The last formula is usually abbreviated as

$$\gamma = \gamma_a \left(1 + f_2 \sin^2\varphi + f_4 \sin^4\varphi \right) , \tag{2–199}$$

so that we have

$$f_2 = -f + \tfrac{5}{2}\,m + \tfrac{1}{2}\,f^2 - \tfrac{26}{7}\,f\,m + \tfrac{15}{4}\,m^2 \,,$$
$$f_4 = -\tfrac{1}{2}\,f^2 + \tfrac{5}{2}\,f\,m \,. \tag{2–200}$$

By substituting

$$\sin^4\varphi = \sin^2\varphi - \tfrac{1}{4}\,\sin^2 2\varphi \,, \tag{2–201}$$

we finally obtain

$$\gamma = \gamma_a \left(1 + f^* \sin^2\varphi - \tfrac{1}{4} f_4 \sin^2 2\varphi\right), \tag{2–202}$$

where

$$f^* = \frac{\gamma_b - \gamma_a}{\gamma_a} = f_2 + f_4 \tag{2–203}$$

is the "gravity flattening".

Coefficients of spherical harmonics

Equation (2–167) for the principal moments of inertia yields at once

$$\frac{C-A}{M\,E^2} = \frac{1}{3} - \frac{2}{45}\,\frac{m\,e'}{q_0}\,. \tag{2–204}$$

Expanding q_0 by means of (2–183), we find

$$\frac{C-A}{M\,E^2} = \frac{1}{e'^2}\left(\tfrac{1}{3}\,e'^2 - \tfrac{1}{3}\,m - \tfrac{2}{7}\,e'^2 m\right). \tag{2–205}$$

Substituting this into (2–170) yields

$$\begin{aligned} -C_{20} = J_2 = \frac{C-A}{M\,E^2} &= \tfrac{1}{3}\,e'^2 - \tfrac{1}{3}\,m - \tfrac{1}{3}\,e'^4 + \tfrac{1}{21}\,e'^2 m \\ &= \tfrac{2}{3}\,f - \tfrac{1}{3}\,m - \tfrac{1}{3}\,f^2 + \tfrac{2}{21}\,f\,m\,, \end{aligned} \tag{2–206}$$

$$-C_{40} = J_4 = -\tfrac{1}{5}\,e'^4 + \tfrac{2}{7}\,e'^2 m = -\tfrac{4}{5}\,f^2 + \tfrac{4}{7}\,f\,m\,. \tag{2–207}$$

The higher C or J, respectively, are already of an order of magnitude that we have neglected.

Gravity above the ellipsoid

Denoting the height above the ellipsoid as ellipsoidal height h, then, in case of a small height, the normal gravity γ_h at this height can be expanded into a series in terms of h:

$$\gamma_h = \gamma + \frac{\partial\gamma}{\partial h}\,h + \frac{1}{2}\,\frac{\partial^2\gamma}{\partial h^2}\,h^2 + \cdots, \tag{2–208}$$

where γ and its derivatives are referred to the ellipsoid, where $h = 0$.

The first derivative $\partial\gamma/\partial h$ may be obtained by applying Bruns' formula (2–147) together with (2–148) to the ellipsoidal height h (instead of H):

$$\frac{\partial\gamma}{\partial h} = -\gamma\left(\frac{1}{M} + \frac{1}{N}\right) - 2\omega^2\,, \tag{2–209}$$

where M, N are the principal radii of curvature of the ellipsoid, defined by (2–149). Since

$$\begin{aligned}\frac{1}{M} &= \frac{b}{a^2}\left(1+e'^2\cos^2\varphi\right)^{3/2} = \frac{b}{a^2}\left(1+\tfrac{3}{2}\,e'^2\cos^2\varphi\,\cdots\right),\\ \frac{1}{N} &= \frac{b}{a^2}\left(1+e'^2\cos^2\varphi\right)^{1/2} = \frac{b}{a^2}\left(1+\tfrac{1}{2}\,e'^2\cos^2\varphi\,\cdots\right),\end{aligned} \tag{2–210}$$

we have

$$\frac{1}{M}+\frac{1}{N} = \frac{b}{a^2}\left(2+2e'^2\cos^2\varphi\right) = \frac{2b}{a^2}\left(1+2f\,\cos^2\varphi\right). \tag{2–211}$$

Here we have limited ourselves to terms linear in f, since the elevation h is already a small quantity. Thus, we find from (2–209) after simple manipulations:

$$\frac{\partial\gamma}{\partial h} = -\frac{2\gamma}{a}\left(1+f+m-2f\,\sin^2\varphi\right). \tag{2–212}$$

The second derivative $\partial^2\gamma/\partial h^2$ may be taken from the spherical approximation, obtained by neglecting e'^2 or f:

$$\gamma = \frac{GM}{a^2}\,,\quad \frac{\partial\gamma}{\partial h} = \frac{\partial\gamma}{\partial a} = -\frac{2GM}{a^3}\,,\quad \frac{\partial^2\gamma}{\partial h^2} = \frac{\partial^2\gamma}{\partial a^2} = \frac{6GM}{a^4}\,, \tag{2–213}$$

so that

$$\frac{\partial^2\gamma}{\partial h^2} = \frac{6\gamma}{a^2}\,. \tag{2–214}$$

Thus we obtain

$$\gamma_h = \gamma\left[1-\frac{2}{a}\left(1+f+m-2f\sin^2\varphi\right)h+\frac{3}{a^2}\,h^2\right]. \tag{2–215}$$

Using Eq. (2–198) for γ, we may also write the difference $\gamma_h-\gamma$ in the form

$$\gamma_h-\gamma = -\frac{2\gamma_a}{a}\left[1+f+m+\left(-3f+\tfrac{5}{2}\,m\right)\sin^2\varphi\right]h+\frac{3\gamma_a}{a^2}\,h^2\,. \tag{2–216}$$

The symbol γ_h denotes the normal gravity for a point at latitude φ, situated at height h above the ellipsoid; γ is the gravity at the ellipsoid itself, for the same latitude φ, as given by (2–202) or equivalent formulas.

Second-order series developments for the *inner* gravity field are found in Moritz (1990: Chap. 4); this is the main reason for such a development here, because today one uses the closed formulas wherever possible.

2.11 Reference ellipsoid – numerical values

Some history

The reference ellipsoid and its gravity field are completely determined by four constants. Before the satellite era, one took the following four parameters:

$$\begin{aligned} a &\ \dots \ \text{semimajor axis}\,, \\ f &\ \dots \ \text{flattening}\,, \\ \gamma_a &\ \dots \ \text{equatorial gravity}\,, \\ \omega &\ \dots \ \text{angular velocity}\,. \end{aligned} \tag{2–217}$$

The values best known and most widely used have been those of the *International Ellipsoid*:

$$\begin{aligned} a &= 6\,378\,388.000 \text{ m}\,, \\ f &= 1/297.000\,, \\ \gamma_a &= 978.049\,000 \text{ gal}\,, \\ \omega &= 0.729\,211\,51 \cdot 10^{-4} \text{ s}^{-1}\,. \end{aligned} \tag{2–218}$$

The geometric parameters a and f were determined by Hayford in 1909 from isostatically reduced astrogeodetic data in the United States. They were adopted for the International Ellipsoid by the assembly of the International Association of Geodesy (IAG) at Madrid in 1924. The equatorial gravity value γ_a was computed by Heiskanen (1924, 1928) from isostatically reduced gravity data. The corresponding *international gravity formula*,

$$\gamma = 978.0490\,(1 + 0.005\,2884 \sin^2\varphi - 0.000\,0059 \sin^2 2\varphi) \text{ gal}\,, \tag{2–219}$$

was adopted by the assembly of IAG at Stockholm in 1930; whose coefficients were computed from the assumed values for a, f, γ_a, ω by Cassinis (1930) using Eqs. (2–200), (2–202), (2–203).

All parameters of the International Ellipsoid and its gravity field can be computed from (2–218) to any desired degree of accuracy, which merely expresses the inner consistency. In this way, we find (rounded values)

$$\begin{aligned} b &= 6\,356\,912 \text{ m}\,, \\ E &= 522\,976 \text{ m}\,, \\ e'^2 &= 0.006\,7682\,, \\ m &= 0.003\,4499\,. \end{aligned} \tag{2–220}$$

For the constants in the spherical-harmonic expansion of the normal gravity field, we find the values

$$\begin{aligned} -C_{20} &= J_2 = \frac{C-A}{M\,a^2} = 0.001\,0920\,, \\ -C_{40} &= J_4 = -0.000\,002\,43\,. \end{aligned} \tag{2–221}$$

The change of normal gravity with elevation is given by the formula (2–216), which for the International Ellipsoid becomes

$$\gamma_h = \gamma - (0.308\,77 - 0.000\,45\,\sin^2\varphi)\,h + 0.000\,072\,h^2\,, \tag{2–222}$$

where γ_h and γ are measured in gal, and h is the elevation in kilometer.

Although the International Ellipsoid can no longer be considered the closest approximation of the earth by an ellipsoid, it may still be used as a reference ellipsoid for geodetic purposes. An official change of a reference system must be very carefully considered because a large amount of data may be referred to such a system.

The eastern countries have used the ellipsoid of Krassowsky:

$$\begin{aligned} a &= 6\,378\,245\ \text{m}\,, \\ f &= 1/298.3\,. \end{aligned} \tag{2–223}$$

Contemporary data

After the start of Sputnik, the first artificial satellite, in 1957, the International Astronomical Union, in 1964, adopted a new set of constants, among them $a = 6\,378\,160$ m and $f = 1/298.25$. The value of a, which is considerably smaller than that for the International Ellipsoid, incorporates astrogeodetic determinations; the change in the value of J_2, and consequently of f, is due to the results from artificial satellites.

In 1967, these values were taken by the International Union of Geodesy and Geophysics (IUGG) as the *Geodetic Reference System 1967.*

This decision was soon seen to be wrong; especially the value of a was recognized to be too large: now we believe to be on the order of 6 378 137 m, the value of the Geodetic Reference System 1980 (GRS 1980) and, based on it, the World Geodetic System 1984 (WGS 84). More details of these two systems are given below.

Geodetic Reference System 1980 (GRS 1980)

The GRS 1980 has been adopted at the XVII General Assembly of the IUGG in Canberra, December 1979, by Resolution No. 7. Inherently, this resolution *recognizing* that the Geodetic Reference System 1967 adopted at the XIV General Assembly of IUGG, Lucerne, 1967, no longer represents the size, shape, and gravity field of the earth to an accuracy adequate for many geodetic, geophysical, astronomical, and hydrographic applications and *considering* that more appropriate values are now available, *recommends* that the Geodetic Reference System 1967 be replaced by the new Geodetic Reference System 1980 which is also based on the theory of the geocentric equipotential ellipsoid. The four defining parameters of the GRS 1980 are given in

Table 2.1. Defining parameters of the GRS 1980

Parameter and value	Description
$a = 6\,378\,137$ m	semimajor axis of the ellipsoid
$GM = 3\,986\,005 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$	geocentric gravitational constant of the earth (including the atmosphere)
$J_2 = 108\,263 \cdot 10^{-8}$	dynamical form factor of the earth (excluding the permanent tidal deformation)
$\omega = 7\,292\,115 \cdot 10^{-11}\ \mathrm{rad\,s^{-1}}$	angular velocity of the earth

Table 2.1. Note that these parameters, as given in the table, are defined as exact! Note also that GM, the "geocentric gravitational constant" of the earth, may also more figuratively be denoted as "product of the (Newtonian) gravitational constant and the earth's mass".

On the basis of these defining parameters and by the computational formulas given in Moritz (1980 b), the geometrical and physical constants of Table 2.2 may be derived.

The GRS 1980 is still (2005) valid as the official reference system of the IUGG and it forms the fundamental basis of the WGS 84.

World Geodetic System 1984 (WGS 84)

As just mentioned, the WGS 84 may be regarded as a descendant of the GRS 1980. Due to its still increasing importance, we consider it appropriate to describe the WGS 84 in some more detail.

Following the National Imagery and Mapping Agency (2000) of the USA, the *definition* of the WGS 84 may be described in the following way. The WGS 84 is a Conventional Terrestrial Reference System (CTRS). The definition of this coordinate system follows the criteria as outlined by the International Earth Rotation Service (IERS). The criteria for this system are the following:

- it is geocentric, the center of mass being defined for the whole earth including oceans and atmosphere;
- its scale is that of the local earth frame, in the meaning of a relativistic theory of gravitation;
- its orientation was initially given by the Bureau International de l'Heure (BIH) orientation of 1984.0;
- its time evolution in orientation will create no residual global rotation with regards to the crust.

Table 2.2. GRS 1980 derived constants

Parameter and value	Description
Geometrical constants	
$b = 6\,356\,752.3141$ m	semiminor axis of the ellipsoid
$E = 521\,854.0097$ m	linear eccentricity
$c = 6\,399\,593.6259$ m	polar radius of curvature
$e^2 = 0.006\,694\,380\,022\,90$	first eccentricity squared
$e'^2 = 0.006\,739\,496\,775\,48$	second eccentricity squared
$f = 0.003\,352\,810\,681\,18$	flattening
$1/f = 298.257\,222\,101$	reciprocal flattening
Physical constants	
$U_0 = 62\,636\,860.850$ m^2 s^{-2}	normal potential at the ellipsoid
$J_4 = -0.000\,002\,370\,912\,22$	spherical-harmonic coefficient
$J_6 = 0.000\,000\,006\,083\,47$	spherical-harmonic coefficient
$J_8 = -0.000\,000\,000\,014\,27$	spherical-harmonic coefficient
$m = 0.003\,449\,786\,003\,08$	$m = \omega^2 a^2 b/(GM)$
$\gamma_a = 9.780\,326\,7715$ m s^{-2}	normal gravity at the equator
$\gamma_b = 9.832\,186\,3685$ m s^{-2}	normal gravity at the pole

The WGS 84 is a right-handed, earth-fixed orthogonal coordinate system. The origin and axes are defined in the following way:

- Origin: earth's center of mass.
- Z-axis: the direction of the IERS Reference Pole (IRP); this direction corresponds to the direction of the BIH Conventional Terrestrial Pole (CTP) (epoch 1984.0). In other terms, the Z-axis is, by convention, identical to the mean position of the earth's rotational axis.
- X-axis: intersection of the IERS Reference Meridian (IRM) and the plane passing through the origin and normal to the Z-axis; the IRM is coincident with the BIH Zero Meridian (epoch 1984.0); in other terms, the X-axis is associated with the mean Greenwich meridian.
- Y-axis: this axis completes a right-handed, earth-centered-earth-fixed (ECEF) orthogonal coordinate system.

The WGS 84 origin also serves as the geometric center of the WGS 84 ellipsoid and the Z-axis serves as the rotational axis of this ellipsoid of revolution.

This completes the definition of the WGS 84 as given in National Imagery and Mapping Agency (2000). Note that the *definition* of the WGS 84 CTRS has not changed in any fundamental way.

Reference frames: WGS 84 and ITRF

Now we need the distinction between *definition* and *realization*. When using the term "coordinate system" or "reference system", then this implies the definition only; however, when using the term "coordinate frame", then a realization is implied (Mueller 1985). So far, we have only given a definition of the WGS 84; therefore, we ought to denote this as WGS 84 CTRS. Now we consider a realization and, therefore, use the term "coordinate frame".

Following closely National Imagery and Mapping Agency (2000) and Hofmann-Wellenhof et al. (2001: Sect. 3.2.1), an example of a terrestrial reference frame is – on the basis of the previous definition – the WGS 84 reference frame (often simply denoted as WGS 84 – as we will also do). Associated to this frame is a geocentric ellipsoid of revolution, originally defined by the four parameters (1) semimajor axis a, (2) normalized second degree zonal gravitational coefficient $\bar{C}_{20}$, (3) truncated angular velocity of the earth ω, and (4) earth's gravitational constant G. This frame has been used for GPS since 1987.

Another example for a terrestrial reference frame is the one produced by the IERS and is called International Terrestrial Reference Frame (ITRF) (McCarthy 1996). The definition of the axes is analogous to the WGS 84, i.e., the Z-axis is defined by the IERS Reference Pole (IRP) and the X-axis lies in the IERS Reference Meridian (IRM); however, the realization differs! The ITRF is realized by a number of terrestrial sites where temporal effects (plate tectonics, tidal effects) are also taken into account. Thus, ITRF is regularly updated (almost every year) and the acronym is supplemented by the last two digits of the last year whose data were used in the formation of the frame, e.g., ITRF89, ITRF90, ITRF91, ITRF92, ITRF93, ITRF94, ITRF95, ITRF96, ITRF97, or the full designation of the year, e.g., ITRF2000.

The comparison of the original WGS 84 and ITRF revealed remarkable differences (Malys and Slater 1994):

1. The WGS 84 was established through Doppler observations from the TRANSIT satellite system, while ITRF is based on Satellite Laser Ranging (SLR) and Very Long Baseline Interferometry (VLBI) observations. The accuracy of the TRANSIT reference stations was estimated to be in the range of 1 to 2 meters, while the accuracy of the ITRF reference stations is at the centimeter level.
2. The numerical values for the original defining parameters differ from those in the ITRF. The only significant difference, however, was in the earth's gravitational constant $G_{\mathrm{WGS}} - G_{\mathrm{ITRF}} = 0.582 \cdot 10^8\,\mathrm{m}^3\,\mathrm{s}^{-2}$, which resulted in measurable differences in the satellite orbits.

On the basis of this information, the former U.S. Defense Mapping Agency

(DMA) has proposed to replace the value of G in the WGS 84 by the standard IERS value and to refine the coordinates of the GPS tracking stations. The revised WGS 84, valid since January 2, 1994, has been given the designation WGS 84 (G 730), where the 'G' indicates that the respective coordinates used were obtained through GPS and the following number 730 indicates the GPS week number when DMA has implemented the refined system.

In 1996, the U.S. National Imagery and Mapping Agency (NIMA) – the successor of DMA – has implemented a revised version of the frame denoted as WGS 84 (G 873) and being valid since September 29, 1996. The frame is realized by monitor stations with refined coordinates. The associated ellipsoid and its gravity field are now defined by the four parameters a, f, GM, ω, which are slightly different compared to the respective ITRF values, e.g., the current WGS 84 (G 873) frame and the ITRF97 show insignificant systematic differences of less than 2 cm. Hence, they are virtually identical.

Note that the refinements applied to the WGS 84 reference frame have reduced the uncertainties in the coordinates of the frame, the uncertainty of the gravitational model, and the uncertainty of the geoid undulations; however, *they have not changed the WGS 84 coordinate system in the sense of definition!*

More general, the relationship between the WGS 84 and the ITRF is characterized by two statements: (1) WGS 84 and ITRF are consistent; (2) the differences between WGS 84 and ITRF are in the centimeter range worldwide (National Imagery and Mapping Agency 2000).

However, if a transformation between reference frames is required, this is accomplished by a datum transformation (see Sect. 5.7).

Numerical values for the WGS 84 (reference frame)

As mentioned at the very beginning of Sect. 2.11, the reference ellipsoid and its gravity field are completely determined by four constants. The current defining parameters for WGS 84 are listed in Table 2.3.

Table 2.3. Defining parameters of the WGS 84

Parameter and value	Description
$a = 6\,378\,137$ m	semimajor axis of the ellipsoid
$f = 1/298.257\,223\,563$	flattening of the ellipsoid
$GM = 3\,986\,004.418 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$	geocentric gravitational constant of the earth (including the atmosphere)
$\omega = 7\,292\,115 \cdot 10^{-11}\ \mathrm{rad\,s^{-1}}$	angular velocity of the earth

Table 2.4. WGS 84 reference ellipsoid derived constants

Parameter and value	Description
Geometrical constants	
$\bar{C}_{20} = -0.484\,166\,774\,985 \cdot 10^{-3}$	normalized second-degree harmonic
$b = 6\,356\,752.3142$ m	semiminor axis of the ellipsoid
$e = 8.181\,919\,084\,2622 \cdot 10^{-2}$	first eccentricity
$e^2 = 6.694\,379\,990\,14 \cdot 10^{-3}$	first eccentricity squared
$e' = 8.209\,443\,794\,9696 \cdot 10^{-2}$	second eccentricity
$e'^2 = 6.739\,496\,742\,28 \cdot 10^{-3}$	second eccentricity squared
$E = 5.218\,540\,084\,2339 \cdot 10^5$	linear eccentricity
$c = 6\,399\,593.6258$ m	polar radius of curvature
$b/a = 0.996\,647\,189\,335$	axis ratio
Physical constants	
$U_0 = 62\,636\,851.7146\ \mathrm{m^2\,s^{-2}}$	normal potential at the ellipsoid
$\gamma_a = 9.780\,325\,3359\ \mathrm{m\,s^{-2}}$	normal gravity at the equator
$\gamma_b = 9.832\,184\,9378\ \mathrm{m\,s^{-2}}$	normal gravity at the pole
$\bar{\gamma} = 9.797\,643\,2222\ \mathrm{m\,s^{-2}}$	mean value of normal gravity
$M = 5.973\,3328 \cdot 10^{24}$ kg	mass of the earth (includes atmosphere)
$m = 0.003\,449\,786\,506\,84$	$m = \omega^2 a^2 b/(GM)$

Some history (even if only some years old) is important here because the parameters selected to originally define the WGS 84 reference ellipsoid were the semimajor axis a, the product of the earth's mass and the gravitational constant GM (also denoted as "geocentric gravitational constant of the earth"), the normalized second-degree zonal gravitational coefficient $\bar{C}_{20}$, and the earth's angular velocity ω. Due to significant refinements of these original defining parameters, the DMA recommended, e.g., a refined value for the GM parameter.

Anyway, a decision was made to retain the original WGS 84 reference ellipsoid values for the semimajor axis $a = 6\,378\,137$ m and for the flattening $f = 1/298.257\,223\,563$. For this reason, the four defining parameters were chosen to be a, f, GM, ω.

Readers who like some confusion may continue right here; otherwise skip this short paragraph. Due to this new choice of the defining parameters, there are in addition two distinct values for the $\bar{C}_{20}$ term, one is dynamically derived and the other geometrically by the defining parameters. The geometric derivation based on the four defining parameters a, f, GM, ω yields $\bar{C}_{20} = -0.484\,166\,774\,985 \cdot 10^{-3}$ which differs from the original value by $7.5015 \cdot 10^{-11}$. For many more details refer to National Imagery and Mapping Agency (2000).

We conclude these considerations by a useful table. Using the four defining parameters, it is possible to derive the more commonly used geometric constants and physical constants (Table 2.4) associated with the WGS 84 reference ellipsoid.

Numerical comparison of GRS 1980 and WGS 84

As mentioned previously, the GRS 1980 is the basis of the WGS 84. However, due to different defining parameters on the one hand and, e.g., a refined value for GM for the WGS 84 on the other hand, numerical differences between the GRS 1980 and the WGS 84 arise. Some of these differences are given in Table 2.5.

Table 2.5. Numerical comparison between GRS 1980 and WGS 84

Parameter	GRS 1980	WGS 84
GM	$3\,986\,005 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$	$3\,986\,004.418 \cdot 10^8\ \mathrm{m^3\,s^{-2}}$
$1/f$	298.257 222 101	298.257 223 563
b	6 356 752.3141 m	6 356 752.3142 m
e^2	0.006 694 380 022 90	0.006 694 379 990 14
e'^2	0.006 739 496 775 48	0.006 739 496 742 28
E	521 854.0097 m	521 854.0084 m
c	6 399 593.6259 m	6 399 593.6258 m
U_0	$62\,636\,860.850\ \mathrm{m^2\,s^{-2}}$	$62\,636\,851.7146\ \mathrm{m^2\,s^{-2}}$
γ_a	$9.780\,326\,7715\ \mathrm{m\,s^{-2}}$	$9.780\,325\,3359\ \mathrm{m\,s^{-2}}$
γ_b	$9.832\,186\,3685\ \mathrm{m\,s^{-2}}$	$9.832\,184\,9378\ \mathrm{m\,s^{-2}}$
m	0.003 449 786 003 08	0.003 449 786 506 84

2.12 Anomalous gravity field, geoidal undulations, and deflections of the vertical

The small difference between the actual gravity potential W and the normal gravity potential U is denoted by T, so that

$$W(x,\,y,\,z) = U(x,\,y,\,z) + T(x,\,y,\,z)\,; \tag{2–224}$$

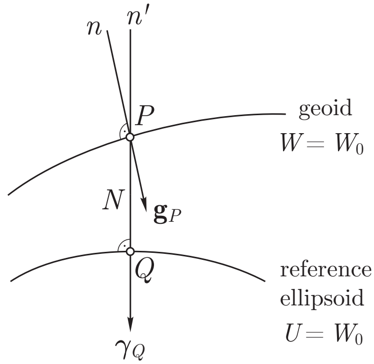

Fig. 2.12. Geoid and reference ellipsoid

T is called the *anomalous potential*, or *disturbing potential*. We compare the geoid

$$W(x, y, z) = W_0 \tag{2–225}$$

with a reference ellipsoid

$$U(x, y, z) = W_0 \tag{2–226}$$

of the same potential $U_0 = W_0$. A point P of the geoid is projected onto the point Q of the ellipsoid by means of the ellipsoidal normal (Fig. 2.12). The distance PQ between geoid and ellipsoid is called the *geoidal height*, or *geoidal undulation*, and is denoted by N. Unfortunately, there is a conflict of notation here. Denoting both the normal radius of curvature of the ellipsoid and the geoidal height by N is well established in geodetic literature. We continue this practice, as there is little chance of confusion.

Consider now the gravity vector $\mathbf{g}$ at P and the normal gravity vector $\boldsymbol{\gamma}$ at Q. The *gravity anomaly vector* $\Delta\mathbf{g}$ is defined as their difference:

$$\Delta\mathbf{g} = \mathbf{g}_P - \boldsymbol{\gamma}_Q \,. \tag{2–227}$$

A vector is characterized by magnitude and direction. The difference in magnitude is the *gravity anomaly*

$$\Delta g = g_P - \gamma_Q \,; \tag{2–228}$$

the difference in direction is the *deflection of the vertical*.

The deflection of the vertical has two components, a north-south component ξ and an east-west component η (Fig. 2.13). As the direction of the vertical is directly defined by the astronomical coordinates latitude Φ and longitude Λ, the components ξ and η can be expressed by them in a simple way. The actual astronomical coordinates of the geoidal point P, which

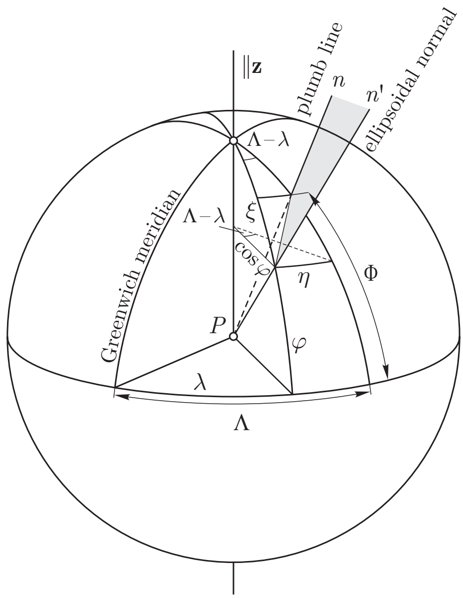

Fig. 2.13. The deflection of the vertical as illustrated by means of a unit sphere with center at P

define the direction of the plumb line n or of the gravity vector $\mathbf{g}$, can be determined by astronomical measurements. The ellipsoidal coordinates (or geodetic coordinates in the sense of geographical coordinates on the ellipsoid) given by the direction of the ellipsoidal normal n' have been denoted by φ and λ – these coordinates should not be confused with the ellipsoidal-harmonic coordinates of Sect. 1.15! It is evident that this λ is identical with the geocentric longitude (and also with the ellipsoidal-harmonic longitude). Thus,

$$\begin{aligned} &\text{geoidal normal } n, \text{ astronomical coordinates } \Phi, \Lambda\,; \\ &\text{ellipsoidal normal } n', \text{ ellipsoidal coordinates } \varphi, \lambda\,. \end{aligned} \tag{2–229}$$

From Fig. 2.13, we read

$$\begin{aligned} \xi &= \Phi - \varphi\,, \\ \eta &= (\Lambda - \lambda)\,\cos\varphi\,. \end{aligned} \tag{2–230}$$

It is also possible to compare the vectors $\mathbf{g}$ and $\boldsymbol{\gamma}$ at the same point P. Then we get the *gravity disturbance vector*

$$\delta\mathbf{g} = \mathbf{g}_P - \boldsymbol{\gamma}_P\,. \tag{2–231}$$

Accordingly, the difference in magnitude is the *gravity disturbance*

$$\delta g = g_P - \gamma_P \,. \tag{2–232}$$

The difference in direction – i.e., the deflection of the vertical – is the same as before, since the directions of γ_P and γ_Q coincide virtually.

The gravity disturbance is conceptually even simpler than the gravity anomaly, but it has not been that important in terrestrial geodesy. The significance of the gravity anomaly is that it is given directly: the gravity g is measured on the geoid (or reduced to it), see Chap. 3, and the normal gravity γ is computed for the ellipsoid.

A very important remark

So far, for historical reasons, much more gravity anomalies Δg are available and are being processed than gravity disturbances δg. By GPS, however, the point P is determined rather than Q. *Therefore, in future, we may expect that δg will become more important than Δg.*

However, mirroring the present state of practice of physical geodesy, we continue mainly to work with Δg. Most statements about Δg will also apply for δg, with obvious modifications, such as with Molodensky's corrections (see Chap. 8), and Stokes' formula will be replaced by Koch's formula (see below in this chapter).

Relations

There are several basic mathematical relations between the quantities just defined. Since

$$U_P = U_Q + \left(\frac{\partial U}{\partial n}\right)_Q N = U_Q - \gamma\, N \,, \tag{2–233}$$

we have

$$W_P = U_P + T_P = U_Q - \gamma\, N + T_P \,. \tag{2–234}$$

Because

$$W_P = U_Q = W_0 \,, \tag{2–235}$$

we find

$$T = \gamma\, N \tag{2–236}$$

(where we have omitted the subscript P on the left-hand side) or

$$N = \frac{T}{\gamma} \,. \tag{2–237}$$

This is the famous *Bruns formula*, which relates the geoidal undulation to the disturbing potential.

Next we consider the gravity disturbance. Since

$$\begin{aligned} \mathbf{g} &= \operatorname{grad} W\,, \\ \boldsymbol{\gamma} &= \operatorname{grad} U\,, \end{aligned} \tag{2–238}$$

the gravity disturbance vector (2–231) becomes

$$\delta\mathbf{g} = \operatorname{grad}\,(W-U) = \operatorname{grad}\,T \equiv \left[\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right]. \tag{2–239}$$

Then

$$g = -\frac{\partial W}{\partial n}\,, \qquad \gamma = -\frac{\partial U}{\partial n'} \doteq -\frac{\partial U}{\partial n}\,, \tag{2–240}$$

because the directions of the normals n and n' almost coincide. Therefore, the gravity disturbance is given by

$$\delta g = g_P - \gamma_P = -\left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n'}\right) \doteq -\left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n}\right) \tag{2–241}$$

or

$$\delta g = -\frac{\partial T}{\partial n}\,. \tag{2–242}$$

Since the elevation h is reckoned along the normal, we may also write

$$\delta g = -\frac{\partial T}{\partial h}\,. \tag{2–243}$$

Comparing (2–242) with (2–239), we see that the gravity disturbance δg, besides being the difference in magnitude of the actual and the normal gravity vector, is also the *normal component of the gravity disturbance vector* $\delta\mathbf{g}$.

We now turn to the gravity anomaly Δg. Since

$$\gamma_P = \gamma_Q + \frac{\partial\gamma}{\partial h}\,N\,, \tag{2–244}$$

we have

$$-\frac{\partial T}{\partial h} = \delta g = g_P - \gamma_P = g_P - \gamma_Q - \frac{\partial\gamma}{\partial h}\,N\,. \tag{2–245}$$

Remembering the definition (2–228) of the gravity anomaly and taking into account Bruns' formula (2–237), we find the following equivalent relations:

$$-\frac{\partial T}{\partial h} = \Delta g - \frac{\partial\gamma}{\partial h}\,N\,, \tag{2–246}$$

$$\Delta g = -\frac{\partial T}{\partial h} + \frac{\partial\gamma}{\partial h}\,N\,, \tag{2–247}$$

$$\Delta g = -\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T \,, \tag{2–248}$$

$$\delta g = \Delta g - \frac{\partial \gamma}{\partial h} N \,, \tag{2–249}$$

$$\delta g = \Delta g - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T \,, \tag{2–250}$$

relating different quantities of the anomalous gravity field.

Another equivalent form is

$$\frac{\partial T}{\partial h} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T + \Delta g = 0 \,. \tag{2–251}$$

This expression has been called the *fundamental equation of physical geodesy*, because it relates the measured quantity Δg to the unknown anomalous potential T. In future, the relation

$$\frac{\partial T}{\partial h} + \delta g = 0 \tag{2–252}$$

may replace it.

It has the form of a partial differential equation. If Δg were known throughout space, then (2–251) could be discussed and solved as a real partial differential equation. However, since Δg is known only along a surface (the geoid), the fundamental equation (2–251) can be used only as a *boundary condition*, which alone is not sufficient for computing T. Therefore, the name "differential equation of physical geodesy", which is sometimes used for (2–251), is rather misleading.

One usually assumes that there are no masses outside the geoid. This is not really true. But neither do we make observations directly on the geoid; we make them on the physical surface of the earth. In reducing the measured gravity to the geoid, the effect of the masses outside the geoid is removed by computation, so that we can indeed assume that all masses are enclosed by the geoid (see Chaps. 3 and 8).

In this case, since the density ϱ is zero everywhere outside the geoid, the anomalous potential T is harmonic there and satisfies Laplace's equation

$$\Delta T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \,. \tag{2–253}$$

This is a true partial differential equation and suffices, if supplemented by the boundary condition (2–251), for determining T at every point outside the geoid. If we write the boundary condition in the form

$$-\frac{\partial T}{\partial n} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial n} T = \Delta g \,, \tag{2–254}$$

where Δg is assumed to be known at every point of the geoid, then we see that a linear combination of T and $\partial T/\partial n$ is given upon that surface. According to Sect. 1.13, the determination of T is, therefore, a *third boundary-value problem of potential theory.* If it is solved for T, then the geoidal height, which is the most important geometric quantity in physical geodesy, can be computed by Bruns' formula (2–237).

Therefore, we may say that the basic problem of physical geodesy, the determination of the geoid from gravity measurements, is essentially a third boundary-value problem of potential theory.

2.13 Spherical approximation and expansion of the disturbing potential in spherical harmonics

The reference ellipsoid deviates from a sphere only by quantities of the order of the flattening, $f \doteq 3 \cdot 10^{-3}$. Therefore, if we treat the reference ellipsoid as a sphere in equations relating quantities of the anomalous field, this may cause a relative error of the same order. This error is usually permissible in N, T, Δg, δg, etc. For instance, the absolute effect of this relative error on the geoidal height is of the order of $3 \cdot 10^{-3}\, N$; since N hardly exceeds 100 m, this error can usually be expected to be less than 1 m.

As a spherical approximation, we have

$$\gamma = \frac{GM}{r^2}\,, \quad \frac{\partial \gamma}{\partial h} = \frac{\partial \gamma}{\partial r} = -2\,\frac{GM}{r^3}\,, \quad \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} = -\frac{2}{r}\,. \tag{2–255}$$

We introduce a mean radius R of the earth. It is often defined as the radius of a sphere that has the same volume as the earth ellipsoid; from the condition

$$\tfrac{4}{3}\,\pi\,R^3 = \tfrac{4}{3}\,\pi\,a^2\,b\,, \tag{2–256}$$

we get

$$R = \sqrt[3]{a^2\,b}\,. \tag{2–257}$$

In a similar way, we may define a mean value of gravity, γ_0, as normal gravity at latitude $\varphi = 45°$ (Moritz 1980b: p. 403). Numerical values of about

$$R = 6371 \text{ km}, \qquad \gamma_0 = 980.6 \text{ gal} \tag{2–258}$$

are usual. Then

$$\begin{aligned} \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} &= -\frac{2}{R}\,, \\ \frac{\partial \gamma}{\partial h} &= -\frac{2\gamma_0}{R}\,. \end{aligned} \tag{2–259}$$

Since the normal to the sphere is the direction of the radius vector r, we have to the same approximation

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial h} = \frac{\partial}{\partial r}\,. \tag{2–260}$$

In Bruns' theorem (2–237) we may replace γ by γ_0, and Eqs. (2–246) through (2–250) and (2–251) become

$$-\frac{\partial T}{\partial h} = \Delta g + \frac{2\gamma_0}{R} N\,, \tag{2–261}$$

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2\gamma_0}{R} N\,, \tag{2–262}$$

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{R} T\,, \tag{2–263}$$

$$\delta g = \Delta g + \frac{2\gamma_0}{R} N\,, \tag{2–264}$$

$$\delta g = \Delta g + \frac{2}{R} T\,, \tag{2–265}$$

$$\frac{\partial T}{\partial r} + \frac{2}{R} T + \Delta g = 0\,. \tag{2–266}$$

The last equation is the spherical approximation of the fundamental boundary condition.

Remark

The meaning of this spherical approximation should be carefully kept in mind. It is used only in equations relating the small quantities T, N, Δg, δg, etc. The reference surface is *never* a sphere in any geometrical sense, but always an ellipsoid. As the flattening f is very small, the ellipsoidal formulas can be expanded into power series in terms of f, and then all terms containing f, f^2, etc., are neglected. In this way one obtains formulas that are rigorously valid for the sphere, but approximately valid for the actual reference ellipsoid as well. However, normal gravity γ in the gravity anomaly $\Delta g = g - \gamma$ must be computed for the ellipsoid to a high degree of accuracy. To speak of a "reference sphere" in space, in any geometric sense, may be highly misleading.

Since the anomalous potential $T = W - U$ is a harmonic function, it can be expanded into a series of spherical harmonics:

$$T(r,\,\vartheta,\,\lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n(\vartheta,\,\lambda)\,. \tag{2–267}$$

$T_n(\vartheta, \lambda)$ is Laplace's surface harmonic of degree n. On the geoid, which as a spherical approximation corresponds to the sphere $r = R$, we have formally

$$T = T(R,\, \vartheta,\, \lambda) = \sum_{n=0}^{\infty} T_n(\vartheta,\, \lambda)\,. \tag{2–268}$$

We need not be concerned with questions of convergence here. Differentiating the series (2–267) with respect to r, we find

$$\delta g = -\frac{\partial T}{\partial r} = \frac{1}{r} \sum_{n=0}^{\infty} (n+1) \left(\frac{R}{r}\right)^{n+1} T_n(\vartheta,\, \lambda)\,. \tag{2–269}$$

On the geoid, where $r = R$, this becomes

$$\delta g = -\frac{\partial T}{\partial r} = \frac{1}{R} \sum_{n=0}^{\infty} (n+1)\; T_n(\vartheta,\, \lambda)\,. \tag{2–270}$$

These series express the gravity disturbance in terms of spherical harmonics.

The equivalent of (2–263) outside the earth is

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}\, T\,. \tag{2–271}$$

Its exact meaning will be discussed at the end of the following section. The substitution of (2–269) and (2–267) into this equation yields

$$\Delta g = \frac{1}{r} \sum_{n=0}^{\infty} (n-1) \left(\frac{R}{r}\right)^{n+1} T_n(\vartheta,\, \lambda)\,. \tag{2–272}$$

On the geoid, this becomes

$$\Delta g = \frac{1}{R} \sum_{n=0}^{\infty} (n-1)\; T_n(\vartheta,\, \lambda)\,. \tag{2–273}$$

This is the spherical-harmonic expansion of the gravity anomaly.

Note that even if the anomalous potential T contains a first-degree spherical term $T_1(\vartheta, \lambda)$, it will in the expression for Δg be multiplied by the factor $1-1=0$, so that Δg can never have a first-degree spherical harmonic – even if T has one.

2.14 Gravity anomalies outside the earth

If a harmonic function H is given at the surface of the earth, then, as a spherical approximation, the values of H outside the earth can be computed by Poisson's integral formula (1–123)

$$H_P = \frac{R}{4\pi} \iint_\sigma \frac{r^2 - R^2}{l^3} H \, d\sigma \, . \tag{2–274}$$

The symbol $\iint_\sigma$ is the usual abbreviation for an integral extended over the whole unit sphere. The meaning of the other notations is read from Fig. 2.14. The value of the harmonic function at the variable surface element $R^2 \, d\sigma$ is denoted simply by H, whereas H_P refers to the fixed point P. Then we get

$$l = \sqrt{r^2 + R^2 - 2Rr \cos\psi} \, . \tag{2–275}$$

The harmonic function H can be expanded into a series of spherical harmonics:

$$H = \left(\frac{R}{r}\right) H_0 + \left(\frac{R}{r}\right)^2 H_1 + \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} H_n \, . \tag{2–276}$$

By omitting the terms of degrees one and zero, we get a new function

$$H' = H - \left(\frac{R}{r}\right) H_0 - \left(\frac{R}{r}\right)^2 H_1 = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} H_n \, . \tag{2–277}$$

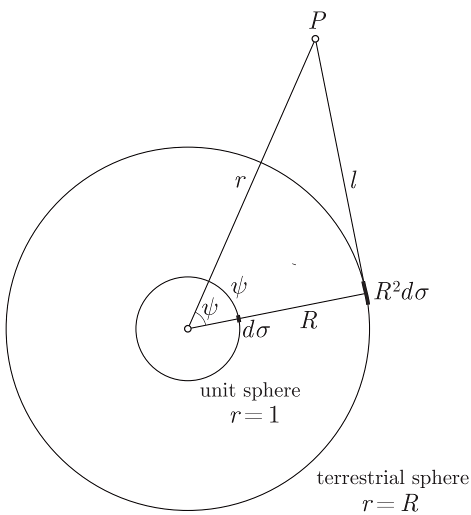

Fig. 2.14. Notations for Poisson's integral and derived formulas

The surface harmonics are given by

$$H_0 = \frac{1}{4\pi} \iint\limits_{\sigma} H \, d\sigma \,, \qquad H_1 = \frac{3}{4\pi} \iint\limits_{\sigma} H \, \cos\psi \, d\sigma \tag{2–278}$$

according to equation (1–89). Hence, we find from (2–277), on expressing H by Poisson's integral (2–274) and substituting the integrals (2–278) for H_0 and H_1, the basic formula

$$H'_P = \frac{1}{4\pi} \iint\limits_{\sigma} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} - \frac{3R}{r^2} \cos\psi \right) H \, d\sigma \,. \tag{2–279}$$

The reason for this modification of Poisson's integral is that the formulas of physical geodesy are simpler if the functions involved do not contain harmonics of degrees zero and one. It is therefore convenient to split off these terms. This is done automatically by the modified Poisson integral (2–279).

We now apply these formulas to the gravity anomalies outside the earth. Equation (2–272) yields at once

$$r \, \Delta g = \sum_{n=0}^{\infty} \left(\frac{R}{r} \right)^{n+1} (n-1) \, T_n(\vartheta, \lambda) \,. \tag{2–280}$$

Just as $T_n(\vartheta, \lambda)$ is a Laplace surface harmonic, so is $(n-1)\,T_n$. Consequently, $r\,\Delta g$, considered as a function in space, can be expanded into a series of spherical harmonics and *is, therefore, a harmonic function.* Hence, we can apply Poisson's formula to $r\,\Delta g$, getting

$$r \, \Delta g_P = \frac{R}{4\pi} \iint\limits_{\sigma} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} - \frac{3R}{r^2} \cos\psi \right) R \, \Delta g \, d\sigma \tag{2–281}$$

or

$$\Delta g_P = \frac{R^2}{4\pi \, r} \iint\limits_{\sigma} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} - \frac{3R}{r^2} \cos\psi \right) \Delta g \, d\sigma \,. \tag{2–282}$$

This is the formula for the computation of gravity anomalies outside the earth from surface gravity anomalies, or for the *upward continuation of gravity anomalies.*

Finally, we discuss the exact meaning of the gravity anomaly δg_P outside the earth. We start with a convenient definition. The level surfaces of the actual gravity potential, the surfaces

$$W = \text{constant} \,, \tag{2–283}$$

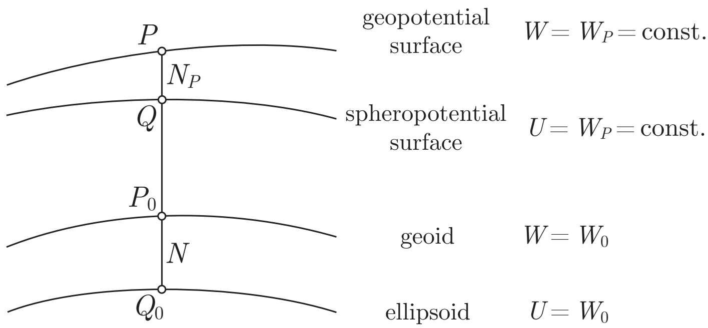

Fig. 2.15. Geopotential and spheropotential surfaces

are often called *geopotential surfaces*; the level surfaces of the normal gravity field, the surfaces

$$U = \text{constant}\,, \tag{2–284}$$

are called *spheropotential surfaces*.

We consider now the point P outside the earth (Fig. 2.15) and denote the geopotential surface passing through it by

$$W = W_P\,. \tag{2–285}$$

There is also a spheropotential surface

$$U = W_P \tag{2–286}$$

of the same constant W_P. The normal plumb line through P intersects this spheropotential surface at the point Q, which is said to correspond to P.

We see that the level surfaces $W = W_P$ and $U = W_P$ are related to each other in exactly the same way as are the geoid $W = W_0$ and the reference ellipsoid $U = W_0$. If, therefore, the gravity anomaly is defined by

$$\Delta g_P = g_P - \gamma_Q\,, \tag{2–287}$$

as in Sect. 2.12, then all derivations and formulas of that section also apply for the present situation, the geopotential surface $W = W_P$ replacing the geoid $W = W_0$, and the spheropotential surface $U = W_P$ replacing the ellipsoid $U = W_0$. This is also the reason why (2–271) applies at P as well as at the geoid.

Note that P in Sect. 2.12 is a point at the geoid, which is denoted by P_0 in Fig. 2.15.

This situation will be taken up again in Chap. 8, in the context of Molodensky's problem.

2.15 Stokes' formula

The basic Eq. (2–271),

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r} T \,, \tag{2–288}$$

can be regarded as a boundary condition only, as long as the gravity anomalies Δg are known only at the surface of the earth. However, by the upward continuation integral (2–282), we are now able to compute the gravity anomalies outside the earth. Thus, our basic equation changes its meaning radically, becoming a real differential equation that can be integrated with respect to r. Note that this is made possible only because T, in addition to the boundary condition, satisfies Laplace's equation $\Delta T = 0$.

Multiplying (2–288) by $-r^2$, we get

$$-r^2 \Delta g = r^2 \frac{\partial T}{\partial r} + 2r\,T = \frac{\partial}{\partial r}(r^2 T) \,. \tag{2–289}$$

Integrating the formula

$$\frac{\partial}{\partial r}(r^2 T) = -r^2 \Delta g(r) \tag{2–290}$$

between the limits ∞ and r, we find

$$r^2 T \Big|_{\infty}^{r} = -\int_{\infty}^{r} r^2 \Delta g(r)\, dr \,, \tag{2–291}$$

where $\Delta g(r)$ indicates that Δg is now a function of r, computed from surface gravity anomalies by means of the formula (2–282). Since this formula automatically removes the spherical harmonics of degrees one and zero from $\Delta g(r)$, the anomalous potential T, as computed from $\Delta g(r)$, cannot contain such terms. Thus, we have

$$T = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n = \frac{R^3}{r^3} T_2 + \frac{R^4}{r^4} T_3 + \cdots \,. \tag{2–292}$$

Therefore,

$$\lim_{r\to\infty} (r^2 T) = \lim_{r\to\infty} \left(\frac{R^3}{r} T_2 + \frac{R^4}{r^2} T_3 + \cdots \right) = 0 \,, \tag{2–293}$$

so that

$$r^2 T \Big|_{\infty}^{r} = r^2 T - \lim_{r\to\infty} (r^2 T) = r^2 T \tag{2–294}$$

and

$$r^2 T = -\int_{\infty}^{r} r^2 \Delta g(r)\, dr \,. \tag{2–295}$$

The fact that r is used both as an integration variable and as an upper limit should not cause any difficulty. Substituting the upward continuation integral (2–282), we get

$$r^2 T = \frac{R^2}{4\pi} \int_\infty^r \left[\iint_\sigma \left(-\frac{r^3 - R^2 r}{l^3} + 1 + \frac{3R}{r} \cos\psi \right) \Delta g \, d\sigma \right] dr \,. \tag{2–296}$$

Interchanging the order of the integrations gives

$$r^2 T = \frac{R^2}{4\pi} \iint_\sigma \left[\int_\infty^r \left(-\frac{r^3 - R^2 r}{l^3} + 1 + \frac{3R}{r} \cos\psi \right) dr \right] \Delta g \; d\sigma \,. \tag{2–297}$$

The integral in brackets can be evaluated by standard methods. The indefinite integral is

$$\int \left(-\frac{r^3 - R^2 r}{l^3} + 1 + \frac{3R}{r} \cos\psi \right) dr$$
$$= \frac{2r^2}{l} - 3l - 3R \cos\psi \, \ln(r - R\cos\psi + l) + r + 3R \cos\psi \, \ln r \,. \tag{2–298}$$

The reader is advised to perform this integration, taking into account (2–275), or at least to check the result by differentiating the right-hand side with respect to r.

For large values of r, we have

$$l = r\left(1 - \frac{R}{r} \cos\psi \, \cdots \right) = r - R \cos\psi \, \cdots \tag{2–299}$$

and, hence, we find that as $r \to \infty$, the right-hand side of the above indefinite integral approaches

$$5R \cos\psi - 3R \cos\psi \, \ln 2 \,. \tag{2–300}$$

If we subtract this from the indefinite integral, we get the definite integral, since infinity is its lower limit of integration. Thus,

$$\int_\infty^r \left(-\frac{r^3 - R^2 r}{l^3} + 1 + \frac{3R}{r} \cos\psi \right) dr$$
$$= \frac{2r^2}{l} + r - 3l - R\cos\psi \left(5 + 3 \ln \frac{r - R\cos\psi + l}{2r} \right) . \tag{2–301}$$

Hence, we obtain Pizzetti's formula

$$T(r, \vartheta, \lambda) = \frac{R}{4\pi} \iint_\sigma S(r, \psi) \, \Delta g \, d\sigma \,, \tag{2–302}$$

where

$$S(r,\psi) = \frac{2R}{l} + \frac{R}{r} - 3\,\frac{R\,l}{r^2} - \frac{R^2}{r^2}\,\cos\psi\left(5 + 3\ln\frac{r - R\,\cos\psi + l}{2r}\right). \quad (2\text{–}303)$$

On the geoid itself, we have $r = R$, and denoting $T(R,\vartheta,\lambda)$ simply by T, we find

$$T = \frac{R}{4\pi}\iint_\sigma \Delta g\,S(\psi)\,d\sigma\,, \quad (2\text{–}304)$$

where

$$S(\psi) = \frac{1}{\sin(\psi/2)} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi\,\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \quad (2\text{–}305)$$

is obtained from $S(r,\psi)$ by setting

$$r = R \quad \text{and} \quad l = 2R\,\sin\frac{\psi}{2}\,. \quad (2\text{–}306)$$

By Bruns' theorem, $N = T/\gamma_0$, we finally get

$$N = \frac{R}{4\pi\,\gamma_0}\iint_\sigma \Delta g\,S(\psi)\,d\sigma\,. \quad (2\text{–}307)$$

This formula was published by G.G. Stokes in 1849; it is, therefore, called *Stokes' formula*, or *Stokes' integral*. It is by far the most important formula of physical geodesy because it performs *to determine the geoid from gravity data*. Equation (2–304) is also called Stokes' formula, and $S(\psi)$ is known as Stokes' function.

Using formula (2–302), which was derived by Pizzetti (1911) and later on by Vening Meinesz (1928), we can compute the anomalous potential T at any point outside the earth. Dividing T by the normal gravity at the given point P (Bruns' theorem), we obtain the separation N_P between the geopotential surface $W = W_P$ and the corresponding spheropotential surface $U = W_P$, which, outside the earth, takes the place of the geoidal undulation N (see Fig. 2.15 and the explanations at the end of the preceding section).

We mention again that these formulas are based on a spherical approximation; quantities of the order of $3 \cdot 10^{-3}\,N$ are neglected. This results in an error of probably less than 1 m in N, which can be neglected for many practical purposes. Sagrebin (1956), Molodenskii et al. (1962: p. 53), Bjerhammar, and Lelgemann have developed higher approximations, which take into account the flattening f of the reference ellipsoid; see Moritz (1980 a: Sect. 39).

We next see from the derivation of Stokes' formula by means of the upward continuation integral (2–282) that it automatically suppresses the harmonic terms of degrees one and zero in T and N. The implications of this will be discussed later. We will see that Stokes' formula in its original form (2–304) and (2–307) only applies for a reference ellipsoid that (1) has the same potential $U_0 = W_0$ as the geoid, (2) encloses a mass that is numerically equal to the earth's mass, and (3) has its center at the center of gravity of the earth. Since the first two conditions are not accurately satisfied by the reference ellipsoids that are in current practical use, and can hardly ever be rigorously fulfilled, Stokes' formula will later be modified for the case of an arbitrary reference ellipsoid.

Finally, T is assumed to be harmonic outside the geoid. This means that the effect of the masses above the geoid must be removed by suitable gravity reductions. This will be discussed in Chaps. 3 and 8.

A bonus application to satellite geodesy

As a somewhat unexpected application, not related to Stokes' formula, we note that Eq. (2–280) can be used to compute gravity anomalies Δg from a satellite-determined spherical-harmonic series of the external gravitational potential V!

2.16 Explicit form of Stokes' integral and Stokes' function in spherical harmonics

We now write Stokes' formula (2–307) more explicitly by introducing suitable coordinate systems on the sphere.

The use of spherical *polar coordinates* with origin at P offers the advantage that the angle ψ, which is the argument of Stokes' function, is one coordinate, the *spherical distance*. The other coordinate is the *azimuth* α, reckoned from north. Their definitions are seen in Fig. 2.16. Denoting by P both a fixed point on the sphere $r = R$ (or in space) and its projection on the unit sphere is common practice and will not cause any trouble.

If P coincides with the north pole, then ψ and α are identical with ϑ and λ. According to Sect. 1.9, the surface element $d\sigma$ is then given by

$$d\sigma = \sin\psi \, d\psi \, d\alpha \, . \tag{2–308}$$

Since all points of the sphere are equivalent, this relation applies for an arbitrary origin P. In the same way, we have

$$\iint_\sigma = \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \, . \tag{2–309}$$

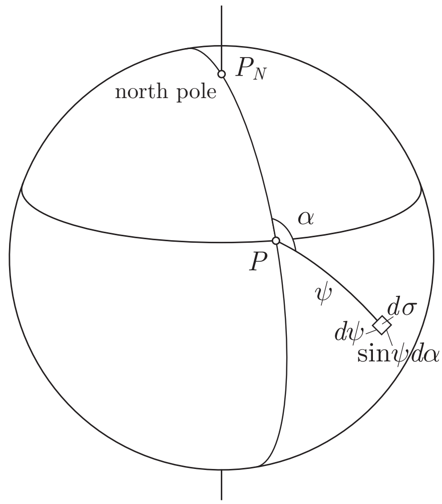

Fig. 2.16. Polar coordinates on the unit sphere

Hence, we find

$$N = \frac{R}{4\pi\,\gamma_0} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \Delta g(\psi,\alpha)\, S(\psi) \sin\psi\; d\psi\; d\alpha \tag{2–310}$$

as an explicit form of (2–307). Performing the integration with respect to α first, we obtain

$$N = \frac{R}{2\gamma_0} \int_{\psi=0}^{\pi} \left[\frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \Delta g(\psi,\alpha)\, d\alpha \right] S(\psi) \sin\psi\; d\psi\,. \tag{2–311}$$

The expression in brackets is the average of Δg along a parallel of spherical radius ψ. We denote this average by $\overline{\Delta g}(\psi)$, so that

$$\overline{\Delta g}(\psi) = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \Delta g(\psi,\alpha)\, d\alpha\,. \tag{2–312}$$

Thus, Stokes' formula may be written

$$N = \frac{R}{\gamma_0} \int_{\psi=0}^{\pi} \overline{\Delta g}(\psi)\, F(\psi)\; d\psi\,, \tag{2–313}$$

where we have introduced

$$\tfrac{1}{2}\, S(\psi)\; \sin\psi = F(\psi)\,. \tag{2–314}$$

The functions $S(\psi)$ and $F(\psi)$ are shown in Fig. 2.17. Alternatively, we may

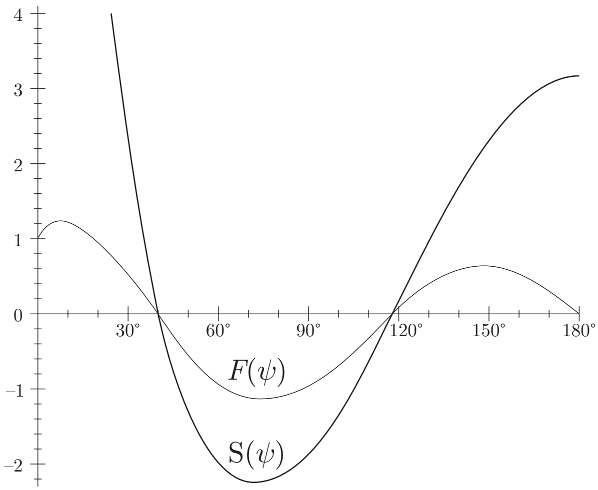

Fig. 2.17. Stokes' functions $S(\psi)$ and $F(\psi)$

use *ellipsoidal coordinates* φ, λ. As a spherical approximation, ϑ is the complement of ellipsoidal latitude:

$$\vartheta = 90^\circ - \varphi\,. \tag{2–315}$$

Hence, we have

$$\iint\limits_{\sigma} d\sigma = \int_{\lambda=0}^{2\pi} \int_{\varphi=-\pi/2}^{\pi/2} \cos\varphi \; d\varphi \; d\lambda\,, \tag{2–316}$$

so that Stokes' formula now becomes

$$N(\varphi,\lambda) = \frac{R}{4\pi\,\gamma_0} \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\pi/2}^{\pi/2} \Delta g(\varphi',\lambda')\, S(\psi) \cos\varphi' \; d\varphi' \; d\lambda'\,, \tag{2–317}$$

where φ, λ are the ellipsoidal coordinates of the computation point and φ', λ' are the coordinates of the variable surface element $d\sigma$. The spherical distance ψ is expressed as a function of these coordinates by

$$\cos\psi = \sin\varphi \; \sin\varphi' + \cos\varphi \; \cos\varphi' \cos(\lambda' - \lambda)\,. \tag{2–318}$$

Stokes' function in terms of spherical harmonics

In Sect. 2.13, Eq. (2–273), we have found

$$\Delta g(\vartheta,\lambda) = \frac{1}{R} \sum_{n=0}^{\infty} (n-1)\; T_n(\vartheta,\lambda)\,. \tag{2–319}$$

We may also directly express $\Delta g(\vartheta, \lambda)$ as a series of Laplace surface spherical harmonics:

$$\Delta g(\vartheta, \lambda) = \sum_{n=0}^{\infty} \Delta g_n(\vartheta, \lambda) \,. \tag{2–320}$$

Comparing these two series yields

$$\Delta g_n(\vartheta, \lambda) = \frac{n-1}{R} \, T_n(\vartheta, \lambda) \quad \text{or} \quad T_n = \frac{R}{n-1} \, \Delta g_n \,, \tag{2–321}$$

so that

$$T = \sum_{n=0}^{\infty} T_n = R \sum_{n=0}^{\infty} \frac{\Delta g_n}{n-1} \,. \tag{2–322}$$

This equation shows again that there must not be a first-degree term in the spherical-harmonic expansion of Δg; otherwise the term $\Delta g_n/(n-1)$ would be infinite for $n = 1$. As usual, we now assume that the harmonics of degrees zero and one are missing. Therefore, we start the summation with $n = 2$.

By Eq. (1–89), we may write

$$\Delta g_n = \frac{2n+1}{4\pi} \iint_{\sigma} \Delta g \, P_n(\cos \psi) \, d\sigma \,, \tag{2–323}$$

so that the preceding formula becomes

$$T = \frac{R}{4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \iint_{\sigma} \Delta g \, P_n(\cos \psi) \, d\sigma \,. \tag{2–324}$$

By interchanging the order of summation and integration, we get

$$T = \frac{R}{4\pi} \iint_{\sigma} \left[\sum_{n=2}^{\infty} \frac{2n+1}{n-1} \, P_n(\cos \psi) \right] \Delta g \, d\sigma \,. \tag{2–325}$$

Comparing this with Stokes' formula (2–304), we find the *expression for Stokes' function in terms of Legendre polynomials* (zonal harmonics):

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \, P_n(\cos \psi) \,. \tag{2–326}$$

In fact, the analytic expression (2–305) of Stokes' function could have been derived somewhat more simply by direct summation of this series, but we believe that the derivation given in the preceding section is more instructive because it also throws sidelights on important related problems such as the "bonus equation" (2–280).

2.17 Generalization to an arbitrary reference ellipsoid

As we have seen, Stokes' formula, in its original form, suppresses the spherical harmonics of degrees zero and one in the anomalous potential T and is, therefore, strictly valid only if these terms are missing. This fact and the condition $U_0 = W_0$ impose restrictions on the reference ellipsoid and on its normal gravity field that are difficult to fulfil in practice.

Therefore, we generalize Stokes' formula so that it will apply to an arbitrary ellipsoid of reference, which must satisfy only the condition that it is so close to the geoid that the deviations of the geoid from the ellipsoid can be treated as linear.

Consider the anomalous potential T at the surface of the earth. Its expression in surface spherical harmonics is given by

$$T(\vartheta, \lambda) = \sum_{n=0}^{\infty} T_n(\vartheta, \lambda) \,. \tag{2–327}$$

By separating the terms of degrees zero and one, we may write

$$T(\vartheta, \lambda) = T_0 + T_1(\vartheta, \lambda) + T'(\vartheta, \lambda) \,, \tag{2–328}$$

where

$$T'(\vartheta, \lambda) = \sum_{n=2}^{\infty} T_n(\vartheta, \lambda) \,. \tag{2–329}$$

In the general case this function T', rather than T itself, is the quantity given by Stokes' formula. It is equal to T only if T_0 and T_1 are missing. Otherwise, we have to add T_0 and T_1 in order to get the complete function T.

The zero-degree term in the spherical-harmonic expansion of the potential is equal to

$$\frac{GM}{r} \,, \tag{2–330}$$

where M is the mass. Hence, the zero-degree term of the anomalous potential $T = W - U$ at the surface of the earth, where $r = R$, is given by

$$T_0 = \frac{G\,\delta M}{R} \,, \tag{2–331}$$

where

$$\delta M = M - M' \tag{2–332}$$

is the difference between the mass M of the earth and the mass M' of the ellipsoid. It would be zero if both masses were equal – but since we do not know the exact mass of the earth, how can we make M' rigorously equal to M?

Subsequently, we will see that the first-degree harmonic can always be assumed to be zero. Under this assumption, we can substitute (2–331) into (2–328) and express T' by the conventional Stokes formula (2–304). Thus we obtain

$$T = \frac{G\,\delta M}{R} + \frac{R}{4\pi} \iint_{\sigma} \Delta g\, S(\psi)\, d\sigma\,. \tag{2–333}$$

This is the generalization of Stokes' formula for T. It holds for an arbitrary reference ellipsoid whose center coincides with the center of the earth.

First-degree terms

The coefficients of the first-degree harmonic in the potential W are, according to (2–85) and (2–87), given by

$$GM\,x_c\,, \quad GM\,y_c\,, \quad GM\,z_c\,, \tag{2–334}$$

where x_c, y_c, z_c are the rectangular coordinates of the earth's center of gravity. For the normal potential U, we have the analogous quantities

$$GM'\,x'_c\,, \quad GM'\,y_c\,, \quad GM'\,z_c\,. \tag{2–335}$$

As x'_c, y'_c, z'_c are very small in any case, these are practically equal to

$$GM\,x'_c\,, \quad GM\,y'_c\,, \quad GM\,z'_c\,. \tag{2–336}$$

The coefficients of the first-degree harmonic in the anomalous potential $T = W - U$ are, therefore, equal to

$$GM\,(x_c - x'_c)\,, \quad GM\,(y_c - y'_c)\,, \quad GM\,(z_c - z'_c)\,. \tag{2–337}$$

They are zero, and *there is no first-degree harmonic $T_1(\vartheta, \lambda)$ if and only if the center of the reference ellipsoid coincides with the center of gravity of the earth.* This is usually assumed.

In the general case, we find from the first-degree term of (2–76), on putting $r = R$ and using the coefficients (2–85) together with (2–87),

$$\begin{aligned} T_1(\vartheta, \lambda) = \frac{GM}{R^2} \Big[&(z_c - z'_c)\, P_{10}(\cos\vartheta) + (x_c - x'_c)\, P_{11}(\cos\vartheta)\, \cos\lambda \\ &+ (y_c - y'_c)\, P_{11}(\cos\vartheta)\, \sin\lambda \Big]. \end{aligned} \tag{2–338}$$

If the origin of the coordinate system is taken to be the center of the reference ellipsoid, then $x'_c = y'_c = z'_c = 0$. With $P_{10}(\cos\vartheta) = \cos\vartheta$, $P_{11}(\cos\vartheta) = \sin\vartheta$, and $GM/R^2 = \gamma_0$ we then obtain the following expression for the first-degree harmonic of T:

$$T_1(\vartheta, \lambda) = \gamma_0 \, (x_c \sin\vartheta \, \cos\lambda + y_c \sin\vartheta \, \sin\lambda + z_c \, \cos\vartheta) \, . \tag{2–339}$$

Dividing by γ_0, we find the first-degree harmonic of the geoidal height:

$$N_1(\vartheta, \lambda) = x_c \sin\vartheta \, \cos\lambda + y_c \sin\vartheta \, \sin\lambda + z_c \cos\vartheta \, . \tag{2–340}$$

Introducing the vector

$$\mathbf{x}_c = [x_c, \; y_c, \; z_c] \tag{2–341}$$

and the unit vector of the direction (ϑ, λ),

$$\mathbf{e} = [\sin\vartheta \cos\lambda, \; \sin\vartheta \, \sin\lambda, \; \cos\vartheta] \, , \tag{2–342}$$

(2–340) may be written as

$$N_1(\vartheta, \lambda) = \mathbf{x}_c \cdot \mathbf{e} \, , \tag{2–343}$$

which is interpreted as the projection of the vector $\mathbf{x}_c$ onto the direction (ϑ, λ).

Hence, if the two centers of gravity do not coincide, then we need only add the first-degree terms (2–339) and (2–340) to the generalized Stokes formula (2–333) and to its analogue for N, respectively, in order to get the most general solution for Stokes' problem, the computation of T and N from Δg. Equation (2–273) shows that *any* value of $T_1(\vartheta, \lambda)$ is compatible with a given Δg field because, for $n = 1$, the quantity $(n-1)\, T_1$ is zero and so T_1, whatever be its value, does not at all enter into Δg.

Hence, the most general solution for T and N contains three arbitrary constants x_c, y_c, z_c, which can, thus, be regarded as the constants of integration for Stokes' problem. In actual practice, one always sets $x_c = y_c = z_c = 0$, thus placing the center of the reference ellipsoid at the center of the earth. This constitutes an essential advantage of the gravimetric determination of the geoid over the astrogeodetic method, where the position of the reference ellipsoid with respect to the center of the earth remains unknown.

Zero-degree terms in N and Δg

Let us first extend Bruns' formula (2–237) to an arbitrary reference ellipsoid. Suppose

$$\begin{aligned} W(x, y, z) &= W_0 \, , \\ U(x, y, z) &= U_0 \end{aligned} \tag{2–344}$$

are the equations of the geoid and the ellipsoid, where in general the constants W_0 and U_0 are different. As in Sect. 2.12, we have, using Fig. 2.12, $W_P = U_Q - \gamma N + T$, but now $U_Q = U_0 \neq W_0 = W_P$, so that

$$\gamma N = T - (W_0 - U_0) . \tag{2–345}$$

Denoting the difference between the potentials by

$$\delta W = W_0 - U_0 , \tag{2–346}$$

we obtain the following simple generalization of Bruns' formula:

$$N = \frac{T - \delta W}{\gamma} . \tag{2–347}$$

We also need the extension of Eqs. (2–246) through (2–250). Those formulas which contain N instead of T are easily seen to hold for an arbitrary reference ellipsoid as well, but the transition from N to T is now effected by means of (2–347). Hence, Eq. (2–247), i.e.,

$$\Delta g = -\frac{\partial T}{\partial h} + \frac{\partial \gamma}{\partial h} N , \tag{2–348}$$

remains unchanged, but (2–248) becomes

$$\Delta g = -\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T - \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} \delta W . \tag{2–349}$$

Therefore, the fundamental boundary condition is now

$$-\frac{\partial T}{\partial h} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} T = \Delta g + \frac{1}{\gamma}\frac{\partial \gamma}{\partial h} \delta W . \tag{2–350}$$

The spherical approximations of these equations are

$$N = \frac{T - \delta W}{\gamma_0} \tag{2–351}$$

and

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{R} T + \frac{2}{R} \delta W \tag{2–352}$$

and

$$-\frac{\partial T}{\partial r} - \frac{2}{R} T = \Delta g - \frac{2}{R} \delta W . \tag{2–353}$$

Relations between T, N, and Δg

By (2–347), we have

$$T = \gamma_0 N + \delta W \,. \tag{2–354}$$

Substituting this into (2–333) and dividing by γ_0, we obtain

$$N = \frac{G\,\delta M}{R\,\gamma_0} - \frac{\delta W}{\gamma_0} + \frac{R}{4\pi\,\gamma_0} \iint_\sigma \Delta g\, S(\psi)\, d\sigma \,. \tag{2–355}$$

This is the generalization of Stokes' formula for N. It applies for an arbitrary reference ellipsoid whose center coincides with the center of the earth.

While formula (2–333) for T contains only the effect of a mass difference δM, the formula (2–355) for N contains, in addition, the potential difference δW. These formulas also show clearly that the simple Stokes integrals (2–304) and (2–307) hold only if $\delta M = \delta W = 0$, that is, if the reference ellipsoid has the same potential as the geoid and the same mass as the earth. Otherwise, they give N and T only up to additive constants: putting

$$N_0 = \frac{G\,\delta M}{R\,\gamma_0} - \frac{\delta W}{\gamma_0} \tag{2–356}$$

and taking into account (2–331), we have

$$T = T_0 + \frac{R}{4\pi} \iint_\sigma \Delta g\, S(\psi)\, d\sigma \,, \tag{2–357}$$

$$N = N_0 + \frac{R}{4\pi\,\gamma_0} \iint_\sigma \Delta g\, S(\psi)\, d\sigma \,. \tag{2–358}$$

Alternative forms of (2–355), which are sometimes useful, are obtained in the following way. Substituting the series (2–268) and (2–270) into (2–352), we get

$$\Delta g(\vartheta, \lambda) = \frac{1}{R} \sum_{n=0}^{\infty} (n-1)\, T_n(\vartheta, \lambda) + \frac{2}{R}\,\delta W \tag{2–359}$$

as the generalization of (2–273). Expanding the function $\Delta g(\vartheta, \lambda)$ into the usual series of Laplace surface spherical harmonics,

$$\Delta g(\vartheta, \lambda) = \sum_{n=0}^{\infty} \Delta g_n(\vartheta, \lambda) \,, \tag{2–360}$$

and comparing the constant terms ($n = 0$) of these two equations, we get

$$-\frac{1}{R}\, T_0 + \frac{2}{R}\,\delta W = \Delta g_0 \,, \tag{2–361}$$

where, by (1–89),

$$\Delta g_0 = \frac{1}{4\pi} \iint_\sigma \Delta g \, d\sigma \,. \tag{2–362}$$

Expressing T_0 by (2–331) in terms of δM, we obtain

$$\Delta g_0 = -\frac{1}{R^2} G\,\delta M + \frac{2}{R}\,\delta W \,. \tag{2–363}$$

The two equations (2–356) for N_0 and (2–363) for Δg_0 can now be solved for δM and δW:

$$\begin{aligned} G\,\delta M &= R\,(R\,\Delta g_0 + 2\gamma_0\,N_0)\,, \\ \delta W &= R\,\Delta g_0 + \gamma_0\,N_0\,. \end{aligned} \tag{2–364}$$

The constant N_0 may be expressed by either of these equations:

$$\begin{aligned} N_0 &= -\frac{R}{2\gamma_0}\,\Delta g_0 + \frac{G\,\delta M}{2\gamma_0\,R}\,, \\ N_0 &= -\frac{R}{\gamma_0}\,\Delta g_0 + \frac{\delta W}{\gamma_0}\,. \end{aligned} \tag{2–365}$$

A final note

A direct consequence of Eq. (2–356) is that N_0 has an immediate geometrical meaning: if a is the equatorial radius (semimajor axis) of the given reference ellipsoid, then

$$a_E = a + N_0 \tag{2–366}$$

is the equatorial radius of an ellipsoid whose normal potential U_0 is equal to the actual potential W_0 of the geoid, and which encloses the same mass as that of the earth, the flattening f remaining the same. The reason is that for such a new ellipsoid E the new $N_0 = 0$ by (2–356) with $\delta M = 0$ and $\delta W = 0$.

A small *additive* constant N_0 is equivalent to a change of *scale* for a nearly spherical earth. To see this, imagine a nearly spherical orange. Increasing the thickness of the peel of an orange everywhere by 1 mm (say) is equivalent to a similarity transformation (uniform increase of the size) of the orange's surface.

So, the usual Stokes formula, without N_0, gives a global geoid that is determined *only up to the scale* which implicitly is contained in N_0. It is, however, *geocentric*, at least in theory, because it contains no spherical harmonic of first degree, $T_1(\vartheta, \lambda)$. It would be exactly geocentric if the earth were covered uniformly by gravity measurements. The scale was formerly determined astrogeodetically, historically by grade measurements dating back to the 18th century (Clairaut, Maupertuis; see Todhunter [1873]). Today, the scale is furnished by satellites (laser, GPS).

2.18 Gravity disturbances and Koch's formula

It is easy to find *Koch's formula*, which is the alternative of Stokes' formula for gravity disturbances δg. We just indicate the road in its general outlines, leaving the reader to generate a four-lane highway.

Compare equations (2–269) and (2–270) with (2–272) and (2–273). We see that the main difference between gravity disturbances δg and gravity anomalies Δg is the spherical harmonic factor $n+1$ and $n-1$, respectively. The other – very small – difference is that we omit in Δg the terms $n=0$ and 1 (see comment after (2–273)), which is not necessary in δg.

Using almost literally the development of Sect. 2.14, we get an equation for δg which is the exact equivalent of (2–282) for Δg. Following the integration in Sect. 2.15, we get a formula of form (2–302)

$$T(r,\vartheta,\lambda) = \frac{R}{4\pi} \iint_\sigma K(r,\psi)\,\delta g\,d\sigma\,, \tag{2–367}$$

and on the sphere $r=R$ we get a formula of form (2–304) which we call *Koch's formula*:

$$T = \frac{R}{4\pi} \iint_\sigma K(\psi)\,\delta g\;d\sigma\,, \tag{2–368}$$

where $K(\psi)$ is the Hotine–Koch function

$$K(\psi) = \frac{1}{\sin(\psi/2)} - \ln\left(1 + \frac{1}{\sin(\psi/2)}\right), \tag{2–369}$$

which is very similar to the Stokes function (2–305). By Bruns' theorem, we finally get

$$N = \frac{R}{4\pi\gamma_0} \iint_\sigma K(\psi)\,\delta g\;d\sigma\,. \tag{2–370}$$

In absolute analogy with (2–326), we have, simply by replacing $n-1$ by $n+1$ and leaving $n=0$ as the lower limit of the sum,

$$K(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1}\,P_n(\cos\psi) \tag{2–371}$$

as the expression of the Koch–Hotine function *in terms of Legendre polynomials* (zonal harmonics). It is really that simple!

A historical remark

This remark is due to Mrs. M.I. Yurkina, Moscow. Mathematically, the above is the solution of Neumann's problem (the second boundary-value problem of potential theory) for the sphere, cf. Sect. 1.13. It is a classical problem of potential theory, with a history of at least 150 years, similarly to Stokes' formula. "Neumann's problem" is named after the mathematician Carl Neumann, who edited his father's (Franz Neumann) lectures from the 1850s (Neumann 1887: see especially p. 275). The external spherical Neumann problem also occurs in Kellogg (1929: p. 247). It is again found in Hotine (1969: pp. 311, 318).

Their basic significance for modern physical geodesy with a known earth surface was recognized and elaborated by Koch (1971). So the present integral formula should perhaps be called F. Neumann−C. Neumann−Kellogg−Hotine−Koch formula. For brevity, we refer to it as *Koch's formula.*

2.19 Deflections of the vertical and formula of Vening Meinesz

Stokes' formula permits the calculation of the geoidal undulations from gravity anomalies. A similar formula for the computation of the deflections of the vertical from gravity anomalies has been given by Vening Meinesz (1928).

Figure 2.18 shows the intersection of geoid and reference ellipsoid with a vertical plane of arbitrary azimuth. If ε is the component of the deflection of the vertical in this plane, then

$$dN = -\varepsilon\, ds \tag{2–372}$$

or

$$\varepsilon = -\frac{dN}{ds}\,; \tag{2–373}$$

the minus sign is a convention, its meaning will be explained later.

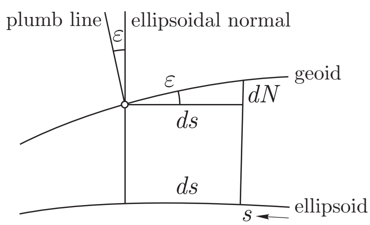

Fig. 2.18. The relation between the geoidal undulation and the deflection of the vertical

In a north-south direction, we have

$$\varepsilon = \xi \quad \text{and} \quad ds = ds_\varphi = R\, d\varphi\,; \tag{2–374}$$

in an east-west direction,

$$\varepsilon = \eta \quad \text{and} \quad ds = ds_\lambda = R \cos\varphi\, d\lambda\,. \tag{2–375}$$

In the formulas for ds_φ and ds_λ, we have again used the spherical approximation; according to (1–30), the element of arc on the sphere $r = R$ is given by

$$ds^2 = R^2\, d\varphi^2 + R^2 \cos^2\!\varphi\, d\lambda^2\,. \tag{2–376}$$

By specializing (2–373), we find

$$\begin{aligned} \xi &= -\frac{dN}{ds_\varphi} = -\frac{1}{R}\,\frac{\partial N}{\partial \varphi}\,, \\ \eta &= -\frac{dN}{ds_\lambda} = -\frac{1}{R\,\cos\varphi}\,\frac{\partial N}{\partial \lambda}\,, \end{aligned} \tag{2–377}$$

which gives the connection between the geoidal undulation N and the components ξ and η of the deflection of the vertical.

As N is given by Stokes' integral, our problem is to differentiate this formula with respect to φ and λ. For this purpose, we use the form (2–317),

$$N(\varphi,\lambda) = \frac{R}{4\pi\,\gamma_0} \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\pi/2}^{\pi/2} \Delta g(\varphi',\lambda')\, S(\psi)\, \cos\varphi'\, d\varphi'\, d\lambda'\,, \tag{2–378}$$

where ψ is defined in (2–318) as a function of φ, λ and φ', λ'.

The integral on the right-hand side of this formula depends on φ and λ only through ψ in $S(\psi)$. Therefore, by differentiating under the integral sign,

$$\frac{\partial N}{\partial \varphi} = \frac{R}{4\pi\,\gamma_0} \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\pi/2}^{\pi/2} \Delta g(\varphi',\lambda')\, \frac{\partial S(\psi)}{\partial \varphi}\, \cos\varphi'\, d\varphi'\, d\lambda' \tag{2–379}$$

is obtained and a similar formula for $\partial N/\partial\lambda$. Here we have

$$\frac{\partial S(\psi)}{\partial \varphi} = \frac{dS(\psi)}{d\psi}\,\frac{\partial \psi}{\partial \varphi}\,, \qquad \frac{\partial S(\psi)}{\partial \lambda} = \frac{dS(\psi)}{d\psi}\,\frac{\partial \psi}{\partial \lambda}\,. \tag{2–380}$$

Differentiating (2–318) with respect to φ and λ, we obtain

$$\begin{aligned} -\sin\psi\,\frac{\partial \psi}{\partial \varphi} &= \cos\varphi\, \sin\varphi' - \sin\varphi\, \cos\varphi'\, \cos(\lambda' - \lambda)\,, \\ -\sin\psi\,\frac{\partial \psi}{\partial \lambda} &= \cos\varphi\, \cos\varphi'\, \sin(\lambda' - \lambda)\,. \end{aligned} \tag{2–381}$$

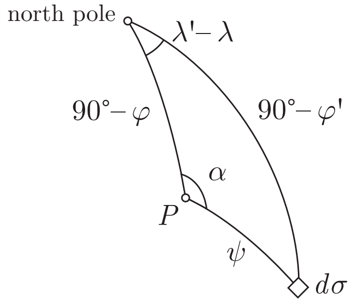

Fig. 2.19. Relation between geographical and polar coordinates on the sphere

We now introduce the azimuth α, as shown in Fig. 2.16. From the spherical triangle of Fig. 2.19 we get, using well-known formulas of spherical trigonometry,

$$\begin{aligned} \sin\psi \, \cos\alpha &= \cos\varphi \, \sin\varphi' - \sin\varphi \, \cos\varphi' \, \cos(\lambda' - \lambda) \,, \\ \sin\psi \, \sin\alpha &= \cos\varphi' \, \sin(\lambda' - \lambda) \,. \end{aligned} \tag{2–382}$$

Substituting these relations into the preceding equations, we find the simple expressions

$$\frac{\partial\psi}{\partial\varphi} = -\cos\alpha \,, \qquad \frac{\partial\psi}{\partial\lambda} = -\cos\varphi \, \sin\alpha \,, \tag{2–383}$$

so that

$$\frac{\partial S(\psi)}{\partial\varphi} = -\frac{dS(\psi)}{d\psi} \cos\alpha \,, \qquad \frac{\partial S(\psi)}{\partial\lambda} = -\frac{dS(\psi)}{d\psi} \cos\varphi \, \sin\alpha \,. \tag{2–384}$$

These are substituted into (2–379) and the corresponding formula for $\partial N/\partial\lambda$ and from equations (2–377) we finally obtain

$$\begin{aligned} \xi(\varphi, \lambda) &= \frac{1}{4\pi\,\gamma_0} \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\pi/2}^{\pi/2} \Delta g(\varphi', \lambda') \, \frac{dS(\psi)}{d\psi} \cos\alpha \, \cos\varphi' \, d\varphi' \, d\lambda' \,, \\ \eta(\varphi, \lambda) &= \frac{1}{4\pi\,\gamma_0} \int_{\lambda'=0}^{2\pi} \int_{\varphi'=-\pi/2}^{\pi/2} \Delta g(\varphi', \lambda') \, \frac{dS(\psi)}{d\psi} \sin\alpha \, \cos\varphi' \, d\varphi' \, d\lambda' \end{aligned} \tag{2–385}$$

or, written in the usual abbreviated form,

$$\begin{aligned} \xi &= \frac{1}{4\pi\,\gamma_0} \iint\limits_{\sigma} \Delta g \, \frac{dS(\psi)}{d\psi} \cos\alpha \, d\sigma \,, \\ \eta &= \frac{1}{4\pi\,\gamma_0} \iint\limits_{\sigma} \Delta g \, \frac{dS(\psi)}{d\psi} \sin\alpha \, d\sigma \,. \end{aligned} \tag{2–386}$$

These are the *formulas of Vening Meinesz*. Differentiating Stokes' function $S(\psi)$, Eq. (2–305), with respect to ψ, we obtain *Vening Meinesz' function*

$$\begin{aligned}\frac{dS(\psi)}{d\psi} &= -\frac{\cos(\psi/2)}{2\sin^2(\psi/2)} + 8\sin\psi - 6\cos(\psi/2) - 3\,\frac{1-\sin(\psi/2)}{\sin\psi} \\ &\quad + 3\sin\psi\,\ln\left[\sin(\psi/2)+\sin^2(\psi/2)\right] .\end{aligned} \tag{2–387}$$

This can be readily verified by using the elementary trigonometric identities. The azimuth α is given by the formula

$$\tan\alpha = \frac{\cos\varphi'\,\sin(\lambda'-\lambda)}{\cos\varphi\,\sin\varphi' - \sin\varphi\,\cos\varphi'\,\cos(\lambda'-\lambda)}\,, \tag{2–388}$$

which is an immediate consequence of (2–382).

The form (2–385) is an expression of (2–386) in terms of ellipsoidal coordinates φ and λ. As with Stokes' formula (Sect. 2.15), we may also use an expression in terms of spherical polar coordinates ψ and α:

$$\begin{aligned}\xi &= \frac{1}{4\pi\,\gamma_0}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi}\Delta g(\psi,\alpha)\,\cos\alpha\,\frac{dS(\psi)}{d\psi}\,\sin\psi\,d\psi\,d\alpha\,, \\ \eta &= \frac{1}{4\pi\,\gamma_0}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\pi}\Delta g(\psi,\alpha)\,\sin\alpha\,\frac{dS(\psi)}{d\psi}\,\sin\psi\,d\psi\,d\alpha\,.\end{aligned} \tag{2–389}$$

The reader can easily verify that these equations give the deflection components ξ and η with the correct sign corresponding to the definition (2–230); see also Fig. 2.13. This is the reason why we introduced the minus sign in (2–373).

We note that the formula of Vening Meinesz is valid as it stands for an arbitrary reference ellipsoid, whereas Stokes' formula had to be modified by adding a constant N_0. If we differentiate the modified Stokes formula with respect to φ and λ, to get Vening Meinesz' formula, then this constant N_0 drops out and we get Eqs. (2–386).

For the practical application of Stokes' and Vening Meinesz' formulas and problems, the reader is referred to Sect. 2.21 and to Chap. 3.

2.20 The vertical gradient of gravity

Bruns' formula (2–40), with $\varrho = 0$,

$$\frac{\partial g}{\partial H} = -2g\,J - 2\omega^2\,, \tag{2–390}$$

cannot be directly applied to determine the gradient $\partial g/\partial H$ because the mean curvature J of the level surfaces is unknown. Therefore, we proceed in the usual way by splitting $\partial g/\partial H$ into a normal and an anomalous part:

$$\frac{\partial g}{\partial H} = \frac{\partial \gamma}{\partial H} + \frac{\partial \Delta g}{\partial H} . \tag{2–391}$$

The normal gradient $\partial\gamma/\partial H$ is given by (2–147) and (2–148). The anomalous part, $\partial\Delta g/\partial H \doteq \partial\Delta g/\partial r$, will be considered now.

Expression in terms of Δg

Equation (2–272) may be written as (note that $r\,\Delta g$ is harmonic and the factor must be 1 for $r = R$)

$$\Delta g(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+2} \Delta g_n(\vartheta, \lambda) . \tag{2–392}$$

By differentiating with respect to r and setting $r = R$, we obtain at sea level:

$$\frac{\partial \Delta g}{\partial r} = -\frac{1}{R} \sum_{n=0}^{\infty} (n+2)\, \Delta g_n = -\frac{1}{R} \sum_{n=0}^{\infty} n\, \Delta g_n - \frac{2}{R}\, \Delta g . \tag{2–393}$$

Now we can apply (1–149), setting $V = \Delta g$ and $Y_n = \Delta g_n$. The result is

$$\frac{\partial \Delta g}{\partial r} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_P}{l_0^3}\, d\sigma - \frac{2}{R}\, \Delta g_P . \tag{2–394}$$

In this equation, Δg_P is referred to the fixed point P at which $\partial\Delta g/\partial r$ is to be computed; l_0 is the spatial distance between the fixed point P and the variable surface element $R^2\, d\sigma$, expressed in terms of the angular distance ψ by

$$l_0 = 2R \sin\frac{\psi}{2} . \tag{2–395}$$

Compare Fig. 1.9 of Sect. 1.14; the element $R^2\, d\sigma$ is at the point P'.

The important integral formula (2–394) expresses the vertical gradient of the gravity anomaly in terms of the gravity anomaly itself. Since the integrand decreases very rapidly with increasing distance l_0, it is sufficient in this formula to extend the integration only over the immediate neighborhood of the point P, as opposed to Stokes' and Vening Meinesz' formulas, where the integration must include the whole earth, if a sufficient accuracy is to be obtained.

Expression in terms of N

By differentiating equation (2–271),

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{r}\,T\,, \tag{2–396}$$

with respect to r, we get

$$\frac{\partial \Delta g}{\partial r} = -\frac{\partial^2 T}{\partial r^2} - \frac{2}{r}\frac{\partial T}{\partial r} + \frac{2}{r^2}\,T\,. \tag{2–397}$$

To this formula we add Laplace's equation $\Delta T = 0$, which in spherical coordinates has the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} - \frac{\tan\varphi}{r^2}\frac{\partial T}{\partial \varphi} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r^2\cos^2\varphi}\frac{\partial^2 T}{\partial \lambda^2} = 0\,; \tag{2–398}$$

see Eq. (1–35), modify by replacing V by T and substitute $\vartheta = 90 - \varphi$. The result, on setting $r = R$, is

$$\frac{\partial \Delta g}{\partial r} = \frac{2}{R^2}\,T - \frac{\tan\varphi}{R^2}\frac{\partial T}{\partial \varphi} + \frac{1}{R^2}\frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{R^2\cos^2\varphi}\frac{\partial^2 T}{\partial \lambda^2}\,. \tag{2–399}$$

Since $T = \gamma_0\,N$, we may also write

$$\frac{\partial \Delta g}{\partial r} = \frac{2\gamma_0}{R^2}\,N - \frac{\gamma_0}{R^2}\tan\varphi\,\frac{\partial N}{\partial \varphi} + \frac{\gamma_0}{R^2}\frac{\partial^2 N}{\partial \varphi^2} + \frac{\gamma_0}{R^2\cos^2\varphi}\frac{\partial^2 N}{\partial \lambda^2}\,, \tag{2–400}$$

where γ_0 is a global mean value as usual. This equation expresses the vertical gradient of the gravity anomaly in terms of the geoidal undulation N and its first and second horizontal derivatives. It can be evaluated by numerical differentiation, using a map of the function N. However, it is less suited for practical application than (2–394) because it requires an extremely accurate and detailed local geoidal map, which is hardly ever available; inaccuracies of N are greatly amplified by forming the second derivatives.

Expression in terms of ξ and η

From equations (2–377), we find

$$\frac{\partial N}{\partial \varphi} = -R\,\xi\,, \qquad \frac{\partial N}{\partial \lambda} = -R\,\eta\,\cos\varphi\,, \tag{2–401}$$

so that

$$\frac{\partial^2 N}{\partial \varphi^2} = -R\,\frac{\partial \xi}{\partial \varphi}\,, \qquad \frac{\partial^2 N}{\partial \lambda^2} = -R\,\frac{\partial \eta}{\partial \lambda}\,\cos\varphi\,. \tag{2–402}$$

Substituting these relations into (2–400) yields

$$\frac{\partial \Delta g}{\partial r} = \frac{2\gamma_0}{R^2} N + \frac{\gamma_0}{R} \xi \tan\varphi - \frac{\gamma_0}{R} \frac{\partial \xi}{\partial \varphi} - \frac{\gamma_0}{R\cos\varphi} \frac{\partial \eta}{\partial \lambda} . \tag{2–403}$$

Introducing local rectangular coordinates x, y in the tangent plane, we have

$$\begin{aligned} R\, d\varphi &= ds_\varphi = dx , \\ R \cos\varphi \, d\lambda &= ds_\lambda = dy , \end{aligned} \tag{2–404}$$

so that (2–403) becomes

$$\frac{\partial \Delta g}{\partial r} = \frac{2\gamma_0}{R^2} N + \frac{\gamma_0}{R} \xi \tan\varphi - \gamma_0 \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right) . \tag{2–405}$$

The first two terms on the right-hand side can be shown to be very small in comparison to the third term; hence, to a sufficient accuracy

$$\frac{\partial \Delta g}{\partial r} = -\gamma_0 \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right) \tag{2–406}$$

may be used. These beautiful formulas express the vertical gradient of the gravity anomaly in terms of the horizontal derivatives of the deflection of the vertical. They can again be evaluated by means of numerical differentiation if a map of ξ and η is available. They are somewhat better suited for practical application than (2–400) because only first derivatives are required.

2.21 Practical evaluation of the integral formulas

Integral formulas such as Stokes' and Vening Meinesz' integrals must be evaluated approximately by summations. The surface elements $d\sigma$ are replaced by small but finite compartments q, which are obtained by suitably subdividing the surface of the earth. Two different methods of subdivision are used:

1. Templates (Fig. 2.20). The subdivision is achieved by concentric circles and their radii. The template is placed on a gravity map of the same scale so that the center of the template coincides with the computation point P on the map. The natural coordinates for this purpose are *polar coordinates* ψ, α with origin at P.

2. Grid lines (Fig. 2.21). The subdivision is achieved by the grid lines of some fixed coordinate system, in particular of *ellipsoidal coordinates* φ, λ. They form rectangular *blocks* – for example, of $10' \times 10'$ or $1^\circ \times 1^\circ$. These blocks are also called squares, although they are usually not squares as defined in plane geometry.

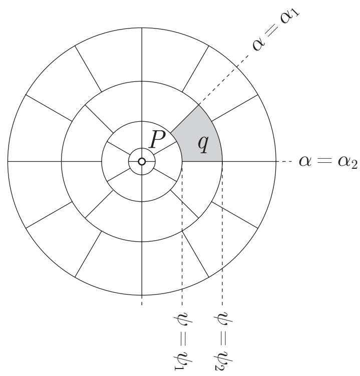

Fig. 2.20. A template

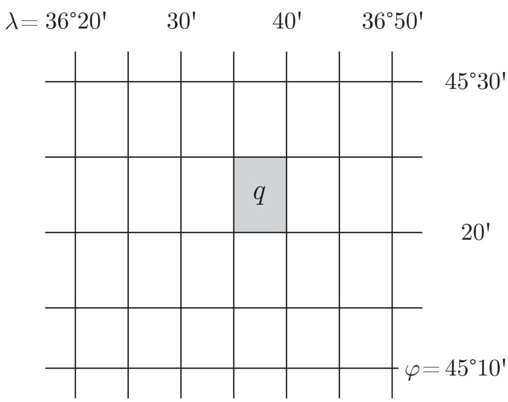

Fig. 2.21. Blocks formed by a grid of ellipsoidal coordinates

The template method is wonderfully easy to understand and to use for theoretical considerations, but completely old-fashioned. Only the gridline method has survived in the computer world.

As a simple and instructive example illustrating the principles of numerical integration consider Stokes' formula

$$N = \frac{R}{4\pi\,\gamma_0} \iint_\sigma \Delta g\, S(\psi)\, d\sigma \tag{2–407}$$

with its explicit forms (2–310) for the template method and (2–317) for the method that uses fixed blocks.

For each compartment q_k, the gravity anomalies are replaced by their average value $\overline{\Delta g}_k$ in this compartment. Hence, the above equation becomes

$$N = \frac{R}{4\pi\,\gamma_0} \sum_k \iint\limits_{q_k} \overline{\Delta g}_k \, S(\psi)\, d\sigma = \frac{R}{4\pi\,\gamma_0} \sum_k \overline{\Delta g}_k \iint\limits_{q_k} S(\psi)\, d\sigma \qquad (2\text{–}408)$$

or

$$N = \sum_k c_k \, \overline{\Delta g}_k\,, \qquad (2\text{–}409)$$

where the coefficients

$$c_k = \frac{R}{4\pi\,\gamma_0} \iint\limits_{q_k} S(\psi)\, d\sigma \qquad (2\text{–}410)$$

are obtained by integration over the compartment q_k; they do not depend on Δg.

If the integrand – in our case, Stokes' function $S(\psi)$ – is reasonably constant over the compartment q_k, it may be replaced by its value $S(\psi_k)$ at the center of q_k. Then we have

$$c_k = \frac{R}{4\pi\,\gamma_0} S(\psi) \iint\limits_{q_k} d\sigma = \frac{S(\psi_k)}{4\pi\,\gamma_0\, R} \iint\limits_{q_k} R^2\, d\sigma\,. \qquad (2\text{–}411)$$

The final integral is simply the area A_k of the compartment and we obtain

$$c_k = \frac{A_k\, S(\psi_k)}{4\pi\,\gamma_0\, R}\,. \qquad (2\text{–}412)$$

The advantage of the template method is its great flexibility. The influence of the compartments near the computation point P is greater than that of the distant ones, and the integrand changes faster in the neighborhood of P. Therefore, a finer subdivision is necessary around P. This can easily be provided by templates. Yet, the method is completely old-fashioned and thus obsolete.

The advantage of the fixed system of blocks formed by a grid of ellipsoidal coordinates lies in the fact that their mean gravity anomalies are needed for many different purposes. These mean anomalies of standard-sized blocks, once they have been determined, can be easily stored and processed by a computer. Also, the same subdivision is used for all computation points, whereas the compartments defined by a template change when the template is moved to the next computation point. The flexibility of the method of standard blocks is limited; however, one may use smaller blocks ($5' \times 5'$, for example) in the neighborhood of P and larger ones ($1^\circ \times 1^\circ$, for example) farther away. With current electronic computation, this method is the only one used in practice. The theoretical usefulness of polar coordinates will be shown now.

Effect of the neighborhood

This issue is interesting and instructive. In the innermost zone, even the template method may pose difficulties if the integrand becomes infinite as $\psi \to 0$. This happens with Stokes' formula, since

$$S(\psi) \doteq \frac{2}{\psi} \tag{2–413}$$

for small ψ. This can be seen from the definition (2–305), because the first term is predominant and is, for small ψ, given by

$$\frac{1}{\sin(\psi/2)} \doteq \frac{1}{(\psi/2)} = \frac{2}{\psi}\,. \tag{2–414}$$

Vening Meinesz' function becomes infinite as well, since to the same approximation,

$$\frac{dS(\psi)}{d\psi} \doteq -\frac{2}{\psi^2}\,. \tag{2–415}$$

In the gradient formula (2–394), the integrand

$$\frac{1}{l_0^3} \doteq \frac{1}{R^3\,\psi^3} \tag{2–416}$$

behaves in a similar way.

Therefore, it may be convenient to split off the effect of this innermost zone, which will be assumed to be a circle of radius ψ_0 around the computation point. For instance, Stokes' integral becomes in this way

$$N = N_i + N_e\,, \tag{2–417}$$

where

$$\begin{aligned} N_i &= \frac{R}{4\pi\,\gamma_0}\int_{\alpha=0}^{2\pi}\int_{\psi=0}^{\psi_0}\Delta g\,S(\psi)\,d\sigma\,,\\ N_e &= \frac{R}{4\pi\,\gamma_0}\int_{\alpha=0}^{2\pi}\int_{\psi=\psi_0}^{\pi}\Delta g\,S(\psi)\,d\sigma\,. \end{aligned} \tag{2–418}$$

The radius ψ_0 of the inner zone corresponds to a linear distance of a few kilometers. Within this distance, we may treat the sphere as a plane, using polar coordinates s, α, where

$$s \doteq R\psi \doteq R\sin\psi \doteq 2R\sin\frac{\psi}{2}\,, \tag{2–419}$$

so that the element of area becomes

$$R^2\,d\sigma = s\,ds\,d\alpha\,. \tag{2–420}$$

It is consistent with this approximation to use (2–413) through (2–416), putting

$$S(\psi) \doteq \frac{2R}{s}\,, \qquad \frac{dS}{d\psi} \doteq -\frac{2R^2}{s^2}\,, \qquad \frac{1}{l_0^3} \doteq \frac{1}{s^3}\,. \tag{2–421}$$

In Stokes' and Vening Meinesz' functions as well, the relative error of these approximations is about 1% for $s = 10$ km, and about 3% for $s = 30$ km. In $1/l_0^3$ it is even less. Hence, the effect of this inner zone on our integral formulas becomes

$$N_i = \frac{1}{2\pi\,\gamma_0} \int_{\alpha=0}^{2\pi} \int_{s=0}^{s_0} \frac{\Delta g}{s}\, s\; ds\; d\alpha\,, \tag{2–422}$$

$$\begin{aligned} \xi_i &= -\frac{1}{2\pi\,\gamma_0} \int_{\alpha=0}^{2\pi} \int_{s=0}^{s_0} \frac{\Delta g}{s^2} \cos\alpha\; s\; ds\, d\alpha\,, \\ \eta_i &= -\frac{1}{2\pi\,\gamma_0} \int_{\alpha=0}^{2\pi} \int_{s=0}^{s_0} \frac{\Delta g}{s^2} \sin\alpha\; s\; ds\, d\alpha\,, \end{aligned} \tag{2–423}$$

$$\left(\frac{\partial \Delta g}{\partial H}\right)_i = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{s_0} \frac{\Delta g - \Delta g_P}{s^3}\, s\; ds\; d\alpha\,. \tag{2–424}$$

In order to evaluate these integrals, we expand Δg into a Taylor series at the computation point P:

$$\Delta g = \Delta g_P + x\, g_x + y\, g_y + \frac{1}{2!}\left(x^2 g_{xx} + 2x\, y\, g_{xy} + y^2 g_{yy}\right) + \cdots\,. \tag{2–425}$$

The rectangular coordinates x, y are defined by

$$x = s\, \cos\alpha\,, \qquad y = s\, \sin\alpha\,, \tag{2–426}$$

so that the x-axis points north. We further have

$$g_x = \left(\frac{\partial \Delta g}{\partial x}\right)_P\,, \qquad g_{xx} = \left(\frac{\partial^2 \Delta g}{\partial x^2}\right)_P\,, \qquad \text{etc.} \tag{2–427}$$

This Taylor series may also be written as

$$\begin{aligned} \Delta g &= \Delta g_P + s\,(g_x \cos\alpha + g_y \sin\alpha) \\ &\quad + \frac{s^2}{2}\left(g_{xx} \cos^2\alpha + 2 g_{xy} \cos\alpha\, \sin\alpha + g_{yy} \sin^2\alpha\right) + \cdots\,. \end{aligned} \tag{2–428}$$

Inserting this into the above integrals, we can easily evaluate them. Performing the integration with respect to α first and noting that

$$\int_0^{2\pi} d\alpha = 2\pi \,,$$

$$\int_0^{2\pi} \sin\alpha \, d\alpha = \int_0^{2\pi} \cos\alpha \, d\alpha = \int_0^{2\pi} \sin\alpha \cos\alpha \, d\alpha = 0 \,, \tag{2–429}$$

$$\int_0^{2\pi} \sin^2\alpha \, d\alpha = \int_0^{2\pi} \cos^2\alpha \, d\alpha = \pi \,,$$

we find

$$N_i = \frac{1}{\gamma_0} \int_0^{s_0} \left[\Delta g_P + \frac{s^2}{4} (g_{xx} + g_{yy}) + \cdots \right] ds \,, \tag{2–430}$$

$$\xi_i = -\frac{1}{2\gamma_0} \int_0^{s_0} (g_x + \cdots) \, ds \,,$$

$$\eta_i = -\frac{1}{2\gamma_0} \int_0^{s_0} (g_y + \cdots) \, ds \,, \tag{2–431}$$

$$\left(\frac{\partial \Delta g}{\partial H} \right)_i = \frac{1}{4} \int_{s=0}^{s_0} (g_{xx} + g_{yy} + \cdots) \, ds \,. \tag{2–432}$$

We now perform the integration over s, retaining only the lowest nonvanishing terms. The result is

$$N_i = \frac{s_0}{\gamma_0} \Delta g_P \,, \tag{2–433}$$

$$\xi_i = -\frac{s_0}{2\gamma_0} g_x \,, \qquad \eta_i = -\frac{s_0}{2\gamma_0} g_y \,, \tag{2–434}$$

$$\left(\frac{\partial \Delta g}{\partial H} \right)_i = \frac{s_0}{4} (g_{xx} + g_{yy}) \,. \tag{2–435}$$

We see that the effect of the innermost circular zone on Stokes' formula depends, to a first approximation, on the value of Δg at P; the effect on Vening Meinesz' formula depends on the first horizontal derivatives of Δg; and the effect on the vertical gradient depends on the second horizontal derivatives.

Note that the contribution of the innermost zone to the total deflection of the vertical has the same direction as the line of steepest inclination of the "gravity anomaly surface", because the plane vector

$$\boldsymbol{\vartheta} = [\xi_i, \ \eta_i] \tag{2–436}$$

is proportional to the horizontal gradient of Δg,

$$\text{grad } \Delta g = [g_x, \ g_y] \,. \tag{2–437}$$

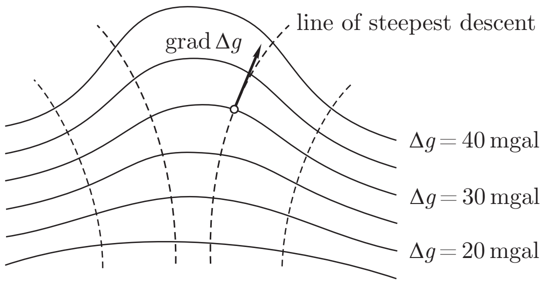

Fig. 2.22. Lines of constant Δg and lines of steepest descent

The direction of grad Δg defines the line of steepest descent (Fig. 2.22). The values of g_x and g_y can be obtained from a gravity map. They are the inclinations of north-south and east-west profiles through P. Values for g_{xx} and g_{yy} may be found by fitting a polynomial in x and y of second degree to the gravity anomaly function in the neighborhood of P.

A remark on accuracy

Deflections of the vertical ξ, η, if combined with astronomical observations of astronomical latitude Φ and astronomical longitude Λ, furnish positions on the reference ellipsoid, expressed by ellipsoidal coordinates

$$\begin{aligned} \varphi &= \Phi - \xi \,, \\ \lambda &= \Lambda - \eta \sec \varphi \,, \end{aligned} \tag{2–438}$$

just as vertical position is obtained by

$$h = H + N \,. \tag{2–439}$$

Unfortunately, to get the same precision for horizontal as for vertical position, is much more difficult, keeping in mind the relation $1'' \cong 30\,\mathrm{m}$ on the earth's surface. So to get an accuracy of 1 m, which is not too difficult with Stokes' formula, means an accuracy better than $0.03''$ in both Φ and ξ (analogously to Λ and η), which is almost impossible to achieve practically.