11 Computational methods

11.1 The remove-restore principle

Let us start with gravity reduction according to the modern view of measuring and calculating the gravity field in principle always *at the earth's surface,* or briefly, *on the ground*, or equivalently, *at point level.* This is used in the sense of Sects. 8.9 and 8.14. More precisely, it is *topographic-isostatic reduction at ground level.*

The most practical way to realize this idea is least-squares collocation, because it automatically works in three-dimensional space, by simply putting the desired topographic height h as parameters for input (measurements: gravity anomalies, deflections of the vertical, etc.) and output (potential T or its functionals to be computed). Symbolically, this means

$$T = \mathcal{L}(\ell) \tag{11–1}$$

or

$$\text{output} = \mathcal{L}(\text{input})\,, \tag{11–2}$$

where $\mathcal{L}$ denotes the linear operation of least-squares collocation (not to be confused with a linear functional L as used, e.g., in Eq. (10–13)).

In Sect. 8.9 we have introduced gravity reduction from the point of view of the modern theory. To repeat, immediately specializing to topographic-isostatic reduction, we have

- measured gravity anomalies Δg at ground level,
- reduced topographic-isostatic anomalies Δg^{c} obtained by *removing* the attraction of the topographic-isostatic masses δg_{TI},
- "co-potential" $T^{\mathrm{c}} = \mathcal{L}(\Delta g^{\mathrm{c}})$ computed by collocation, and
- "real potential" T by *restoring* the "indirect effect" of the topographic-isostatic masses δT_{TI}.

Mathematically this may be written

$$T = \mathcal{L}(\Delta g - \delta g_{\mathrm{TI}}) + \delta T_{\mathrm{TI}}\,. \tag{11–3}$$

This is a reinterpretation of the gravity reduction of Sect. 8.9. It must be correct since if

$$\delta T_{\mathrm{TI}} = \mathcal{L}(\delta g_{\mathrm{TI}}) \tag{11–4}$$

then Eq. (11–3) gives

$$\delta T = \mathcal{L}(\Delta g), \tag{11–5}$$

as it should be.

The same principle works also with deflections of the vertical ξ, η at the earth's surface, both as input data and as output results (Sects. 8.14 and 10.2).

The underlying isostatic model is in principle arbitrary. For practical purposes it should provide a good approximation (small residuals δT) and be computationally convenient.

We see, however, a change of perspective. Collocation is no longer applied to the "real" anomalous gravity field as in (11–5) but to the residual field, removing the field generated by the assumed topographic-isostatic model. The *model* is arbitrary, but the *derived quantities* must be computed in a rigorous consistent fashion. (Consistency for the quantities computed by collocation as guaranteed by a correct covariance propagation; see Sect. 10.2.)

This change of perspective may not seem important because it is just a change of nomenclature: what formerly was importantly called "isostatic anomaly" is now degraded to a miserable "residual". However, the remove-restore principle permits also the use of other approximate fields to remove trends; especially one of the numerous existing "earth (gravity) models" (EM or EGM) consisting of spherical-harmonic expansions of the potential T up to degree 180 or higher.

Therefore, we "remove" from the observations ℓ – gravity anomalies, gravity disturbances, deflections of the vertical, etc. – the effect ℓ_{EM} computed from the earth model used, and after collocation "restore" the effect of the EM on the result. The mathematics is the same as in (11–4) and (11–5):

$$\delta T_{\mathrm{EM}} = \mathcal{L}(\delta \ell_{\mathrm{EM}}) \tag{11–6}$$

and

$$\delta T = \mathcal{L}(\ell). \tag{11–7}$$

We have only slightly generalized from Δg to ℓ.

Now we proceed an important step further. The remove-restore principle has only two requirements:

1. the removed auxiliary potentials must be harmonic, precomputable, and used in a mathematically consistent way: what is removed in the input must be restored in the output;

2. in the usual case of linearity, two or more different auxiliary potentials may be used (removed-restored) simultaneously.

Thus, we use simultaneously the earth model EM for the longer wavelengths and the topographic-isostatic geological model TI for the shorter wavelengths. Since the spherical-harmonic expansions are generalizations, for the sphere, of Fourier series for the circle, we can speak of wavelengths. Denoting the maximum degree of the spherical-harmonic expansion with N, this can be associated with a shortest resolvable wavelength λ according to

$$\lambda = \frac{2\pi}{N} = \frac{360°}{N} . \qquad (11\text{–}8)$$

For an expansion to degree $N = 180$ (say), we have $\lambda = 360°/180 = 2°$, which roughly corresponds to 200 km on a meridian or on the equator. In many cases, the half wavelength $\lambda/2$ is considered (see Seeber 2003: p. 469).

Since EM (approximately) takes care of the long waves up to a certain maximum degree N, it is resonable to represent the remaining short waves from N to infinity. This sequence $N+1, N+2, \ldots, \infty$ will be denoted by CN, where CN is the abbreviation of the "complement" of the sequence from 2 to N.

Thus, we may write for the residuals

$$\begin{aligned} \delta T &= T - T_{\mathrm{EM}}^{N} - T_{\mathrm{TI}}^{CN} , \\ \delta \ell &= \ell - \ell_{\mathrm{EM}}^{N} - \ell_{\mathrm{TI}}^{CN} . \end{aligned} \qquad (11\text{–}9)$$

The collocation procedure will be applied to these residuals.

Remark

As we have noted at the beginning of Sect. 10.2, the remove-restore process aims at removing all known major trends:

- the local topography produces *Bouguer anomalies*,
- the regional features (i.e., their isostatic compensation), in addition to the Bouguer effect, produce *topographic-isostatic anomalies*,
- the global irregularities are expressed by an earth model and lead to what is modestly called the *"residual anomalies"*.

It is clear that what is "removed" before the computation, must be fully "restored" after the computation.

11.2 Geoid in Austria by collocation

Austria is a nice country, and in spite of being small, it has all types of topography: flat, hilly, and alpine, up to an elevation of 3800 m. Thus, beyond being a pleasant place to live, it is an interesting geodetic test area.

The pioneering work has been done by Sünkel (1983). Later work, especially by Sünkel et al. (1987), Kühtreiber (1998, 2002 a, 2002 b), and Erker et al. (2003) has refined, extended and perfected the gravity field in Austria, but the 1983 work is good for an introduction.

Sünkel (1983) used least-squares collocation to calculate the geoid for the main part of Austria from a very good material of deflections of the vertical. Gravity anomalies of a comparable quality were not yet available in 1983. In addition to an isostatic reduction (Sect. 8.14) according to Airy–Heiskanen ($T = 30$ km), he also removed a global trend by means of an earth gravity model, represented by a spherical-harmonic expansion up to a certain degree N. In particular, he used the model of Rapp (1981) with $N = 180$.

After removing the topographic-isostatic trend T_{TI} and this global trend T_{EM}^{N} (remember, EM denotes earth model), there remains a residual anomalous potential δT, given by

$$\delta T = T - T_{\mathrm{TI}} - T_{\mathrm{EM}}^{N} + T_{\mathrm{TI}}^{N} . \tag{11–10}$$

Since the earth model potential T_{EM}^{N} is represented by a spherical-harmonic expansion up to degree N, it may be appropriate to consider, for the isostatic reduction, only the effect for degrees $N > 180$ (or, say, $N > 360$), replacing T_{TI} by

$$T_{\mathrm{TI}}^{CN} = (T_{\mathrm{TI}})_{N>180} = T_{\mathrm{TI}} - T_{\mathrm{TI}}^{N} , \tag{11–11}$$

where T_{TI}^{N} represents a spherical-harmonic expansion for T_{TI} truncated at degree $N = 180$. This explains Eq. (11–11).

The observations $\ell_i = [\xi, \eta, \Delta g]$, which represent linear functionals $L_i T$, are reduced in the same way, obtaining

$$\ell_i - L_i T_{\mathrm{TI}} - L_i T_{\mathrm{EM}}^{N} + L_i T_{\mathrm{TI}}^{N} = L_i \delta T . \tag{11–12}$$

Adding the earth model reduction to the computational procedure outlined at the end of the preceding section, we thus have the flow diagram of Table 11.1.

Data

The topography in Austria is rather varied, with elevations up to 3800 m. The density of astrogeodetic stations was 10 to 20 km; the total number of deflections data used was 521. No gravity anomalies were used in this first computation.

The topographic-isostatic reduction of the deflections of the vertical was made using a rather crude digital terrain model consisting of mean elevations for $20'' \times 20''$ rectangles. It has been obtained by digitizing a map 1 : 500 000.

Table 11.1. From observations to the geoid

$(L_i T)$	observations referred to Geodetic Reference System 1980
$\downarrow$	
reduction TI, EM	$-L_i(T_{\mathrm{TI}} + T^N_{\mathrm{EM}} - T^N_{\mathrm{TI}})$
$\downarrow$	
$(L_i \delta T)$	
$\downarrow$	
collocation	
$\downarrow$	
δT	
$\downarrow$	
inverse reduction TI, EM	$+ \quad (T_{\mathrm{TI}} + T^N_{\mathrm{EM}} - T^N_{\mathrm{TI}})$
$\downarrow$	
T	
$\downarrow$	
$N = (T/\gamma)_0 \,, \quad \zeta = (T/\gamma)_h$	

The standard error of this model is on the order of 100 m. Investigations have shown that, in spite of its poor accuracy, the model is reasonably adequate for reduction of deflections of the vertical; it is, however, totally inadequate for gravity! In fact, the reduction error for ξ, η is approximately proportional to the terrain inclination; it is thus very small if the deflection station is situated in an area of inclination zero. This is the case not only if the station lies in a horizontal plane but also if it lies on the top of a mountain, as most deflection stations do.

Results

It turned out that almost all of the signal (T, N, ζ) comes from the topographic-isostatic model and the $N = 180$ gravity model used. This part, TI + EM, lies between 41.5 m and 49.5 m. The contribution of collocation ($\gamma^{-1} T$) lies between -0.5 m and 1.5 m, after removal of a pronounced trend on the order of 3 m.

The efficiency of topographic-isostatic reduction can also be seen from the fact that it has reduced the variance of the deflections of the vertical in

Austria (the square of the average size of ξ and η) from 30 (arc second)^2 to 5 (arc second)^2.

So we may say that we can determine the Austrian geoid to 1–2 m without measurements (deflections of the vertical) and without collocation, knowing only a topographic map! This is even more surprising since Austria is not particularly well isostatically compensated.

Of considerable interest is the effect of analytical continuation on the isostatically (plus earth model) reduced anomalous potential T_{TI}. It is expressed by the difference $\gamma^{-1} T$ at the earth's surface minus $\gamma^{-1} T$ at sea level. This difference reaches a maximum of 13 cm in the Central Alps and is otherwise positive and negative. In the terminology of the present book, this is the *separation between the real geoid and the harmonic geoid* (Sect. 8.15).

Of the same interest is the difference between the height anomalies ζ ($= \gamma^{-1} T$ at the earth's surface) and the geoidal heights N ($= \gamma^{-1} T$ at sea level). The maximum of 35 cm for $\zeta - N$ is reached at the Grossglockner mountain (the highest peak in Austria, $H = 3797$ m). The results are in excellent agreement with the approximate formula

$$\zeta - N = -(981 \text{ gal})^{-1} \, \Delta g_B \, H \,, \tag{11–13}$$

where Δg_B is the Bouguer anomaly in gal and H is the elevation in the same units as ζ and N. The agreement may easily be verified, since the Bouguer anomalies in the investigated area range from 10 mgal to −170 mgal, corresponding to topographic heights from 200 m to 3000 m (Sünkel 1983: p. 140). In Sünkel et al. (1987: p. 69), the differences $\zeta - N$ for the whole of Austria range between −2 cm and +56 cm.

All this has been computed only from the measured deflections of the vertical. Gravity observations have been included by Kühtreiber (2002 a, 2002 b) and Erker et al. (2003), leading to what might be a "few-centimeter geoid".

Important: the astrogeodetic geoid and the gravimetric geoid are compared and finally combined after systematic trends have been eliminated by Kühtreiber (2002 b) and Erker et al. (2003).

11.3 Molodensky corrections

In Sect. 8.6 we have given a solutions of Molodensky's problem by means of a series obtained on the basis of analytical continuation. It can be written in the form of Eqs. (8–68), (8–69), (8–67),

$$\zeta = \zeta_0 + \zeta_1 + \zeta_2 + \zeta_3 + \cdots \,, \tag{11–14}$$

$$\zeta_i = \frac{R}{4\pi\,\gamma} \iint_\sigma g_i\, S(\psi)\, d\sigma\,, \tag{11–15}$$

$$\Delta g^* = \Delta g + g_1 + g_2 + g_3 + \cdots\,. \tag{11–16}$$

The correction terms g_n are evaluated recursively by

$$g_n = -\sum_{r=1}^{n} z^r\, L_r(g_{n-r})\,, \tag{11–17}$$

starting from

$$g_0 = \Delta g\,. \tag{11–18}$$

Here the operator L_n is also defined recursively:

$$L_n(\Delta g) = n^{-1} L_1\left[L_{n-1}(\Delta g)\right] \tag{11–19}$$

starting with

$$L_1 = L \tag{11–20}$$

with the gradient operator L defined by the integral (8–60), that is,

$$L(f) = \frac{R^2}{2\pi} \iint_\sigma \frac{f - f_Q}{l_0^3}\, d\sigma\,. \tag{11–21}$$

This means: take $g_0 = \Delta g$, where Δg is the free-air anomaly at ground level in the sense of Molodensky, then compute g_1 by (11–17) with $n = 1$, then compute g_2 by (11–17) with $n = 2$ and L_2 by (11–19), then g_3 by (11–17) with $n = 3$ and L_3 by (11–19), etc.

The operator L behaves like differentiation ($L(f) = \frac{\partial \Delta g}{\partial r}$) and, therefore, "roughens" the function f; this means that each successive L becomes rougher and rougher. This is not conducive to the convergence of Molodensky's series unless the original Δg is very smooth, which cannot be assumed in mountainous areas.

In such cases, some smoothing of Δg is inevitable. Numerical analysis is constantly confronted with problems of smoothing, so many techniques of smoothing have been developed such as the sliding average. For evaluating the integral L, fast Fourier methods are available. The problem is to find an appropriate degree of smoothing which makes consecutive corrections g_1, g_2, $g_3, \ldots$ decrease in order to achieve practical convergence without "oversmoothing". At any rate, smoothing must ensure that g_5, g_6, $\ldots$ are practically negligible since they cannot be meaningfully computed because of the inevitable accumulation of round-off errors, which finally tends to producing pure noise.

Table 11.2. Characteristic values in arc seconds for Molodensky corrections ξ_i and η_i for deflections of the vertical until $i = 4$, computed from free-air gravity anomalies

	ξ_1	η_1	ξ_2	η_2	ξ_3	η_3	ξ_4	η_4
min	−2.44	−1.94	−0.92	−0.84	−0.35	−0.24	−0.08	−0.12
max	2.36	3.654	0.88	0.86	0.21	0.20	0.05	0.09
mean	0.19	0.32	−0.02	−0.02	−0.01	−0.01	0.00	0.00
rms	0.90	0.96	0.29	0.27	0.06	0.06	0.02	0.02

Table 11.3. Characteristic values in arc seconds for Molodensky corrections ξ_i and η_i for deflections of the vertical until $i = 4$, computed from isostatic gravity anomalies

	ξ_1	η_1	ξ_2	η_2	ξ_3	η_3	ξ_4	η_4
min	−0.57	−0.36	−0.06	−0.07	−0.01	−0.02	0.00	0.00
max	0.33	0.46	0.09	0.05	0.01	0.01	0.00	0.00
mean	−0.04	0.01	0.00	−0.01	0.00	0.00	0.00	0.00
rms	0.11	0.09	0.02	0.02	0.00	0.00	0.00	0.00

As Kühtreiber (1990) showed in his thorough work, there is no rough-and-ready prescription for finding an optimal smoothing. Trial and error may be the best approach.

Isostatic reduction might be considered a smoothing method on a geophysical basis, cf. Tables 11.2 and 11.3.

Just to give an idea of the order of magnitudes, we take some typical sizes of the Molodensky corrections in high mountains.

We take two tables from Kühtreiber (1990): the following Tables 11.2 and 11.3 are Kühtreiber's Tables (8-3) and (8-6). The gravity data are assumed to be given in a rectangular grid of size $11.25'' \times 18.75''$. A suitable smoothing is presupposed. Much better is, of course, the use of *isostatic reduction*, which should provide a physically meaningful and efficient smoothing. This is shown by Table 11.3.

To provide some contrast and to include also Molodensky corrections for the height anomaly ζ, we quote also Table 11.4 of a somewhat earlier work by Kraiger et al. (1987) (denoted as Table 6.1 there). The values are not directly comparable because test areas and selected methods of integration, smoothing, data density, etc., are different. Still, they lead to interesting

Table 11.4. Comparison of direct numerical integration and fast Fourier transform (FFT): maximum and (arithmetic) mean values of Molodensky corrections ζ_i, ξ_i, η_i for $i = 1, 2$; test area: $46.788° \leq \varphi \leq 46.512°$, $13.438° \leq \lambda \leq 14.646°$, $600\,\mathrm{m} \leq$ topographic height $\leq 2400\,\mathrm{m}$

	ζ_1 [cm]	ξ_1 [″]	η_1 [″]	ζ_2 [cm]	ξ_2 [″]	η_2 [″]	
maximum	40.8	2.0	2.0	0.8	0.2	0.2	direct int.
values	47.6	1.5	1.4	0.7	0.1	0.1	FFT
mean	31.3	0.4	0.4	0.2	0.03	0.03	direct int.
values	36.7	0.3	0.4	0.5	0.02	0.02	FFT

conclusions:

1. The method of Molodensky corrections depends very much on the details of numerical integration (data density, smoothing, etc.).

2. The corrections decrease for increasing $i = 1, 2, 3, \ldots$. This is what they have to do. Higher corrections may be expected finally to consist of “pure noise” because of general roughening and increasing round-off errors, so that the question of convergence becomes practically as well as theoretically meaningless: higher terms must simply be put equal to zero by higher force.

3. The Molodensky correction ζ_1 may reach a few decimeters, ζ_2 and higher-order terms might frequently be negligible.

4. At the end of Sects. 2.21 and 8.8, we have remarked a curious phenomenon. Using the same data, gravimetric methods seem to furnish the vertical position (expressed by ζ or N) roughly by one order of magnitude better than the horizontal position (as expressed by ξ, η). If we take the old astronomer’s rule that $1'' \cong 30\,\mathrm{m}$ in position, then $1\,\mathrm{m}$ corresponds to $0.03''$. Assume that we get $1\,\mathrm{m}$ in vertical position and wish to get the same accuracy for horizontal position. This would mean that we have to get the astronomical measurements Φ, Λ and the deflections of the vertical ξ, η with better than $0.03''$. This also seems to apply with the order of magnitude of the Molodensky corrections, where a Molodensky correction $\zeta_1 = 0.41\,\mathrm{m}$ comes along with a $\xi_1 = \eta_1 = 2''$, which corresponds to $60\,\mathrm{m}$.

   In this sense, gravimetry is weaker by one order of magnitude in determining the horizontal than the vertical position. This is an admittedly one-sided perspective, but it was used against scientists who claimed, still around 1960, that the gravimetric method was able to do everything that satellite geodesy could. With GPS now we know better, and without ideological scruples we combine satellite data with terrestrial gravity.

(A second perspective of astronomical observations is the astrogeodetic geoid determination. Here the accuracy of astronomy is sufficient; cf. Sec. 5.14.)

Final remark

The computation of Molodensky reductions is heavy work. So in mountainous areas, least-squares collocation is definitely preferable to integration, except for certain test computations (Sideris 1987, 1990).

Collocation also permits comparison and combination of astrogeodetic and gravimetric data; a key paper is Kühtreiber (2002 b).

All this, however, builds on the fundamental ideas of M.S. Molodensky. In his landmark publication (Krarup 1969) one clearly sees the transition from Molodensky's problem to least-squares collocation.

11.4 The geoid on the internet

The International Association of Geodesy (IAG) has a very active International Geoid Service (IGeS–IAG). Before you try to compute your own geoid, look at www.iges.polimi.it to see what is available there. You can find global and regional geoids, data, software, references, plans for future work, etc. We particularly mention the geoid repositories:

- www.iges.polimi.it/index/geoid_repo/global_models.htm ,
- www.iges.polimi.it/index/geoid_repo/regional_models.htm .

In the latter file you can find:

- USA gravimetric Geoid 1996 (Dru Smith),
- European Geoid/Quasigeoid EGG97 (H. Denker),
- Austrian Geoid 1996 (H. Sünkel).

Other important internet addresses:

- International Gravity Bureau (Toulouse):
  http://bgi.cnes.fr8110/bgi_a.html ,
- International Association of Geodesy:
  www.iag_aig.org .