1 Fundamentals of potential theory

1.1 Attraction and potential

The purpose in this preparatory chapter is to present the fundamentals of potential theory, including spherical and ellipsoidal harmonics, in sufficient detail to assure a full understanding of the later chapters. Our intent is to explain the meaning of the theorems and formulas, avoiding long derivations that can be found in any textbook on classical (before 1950) potential theory; we recommend Kellogg (1929). A simple rather than completely rigorous presentation is offered in our book.

Nevertheless, the reader might consider this chapter perhaps more difficult and abstract than other parts of the book. Since later practical applications will give concrete meaning to the topics of the present chapter, the reader may wish to read it only cursorily at first and return to it later when necessary.

According to Newton's law of gravitation, two points with masses m_1, m_2, separated by a distance l, attract each other with a force

$$F = G\,\frac{m_1 m_2}{l^2}\,. \tag{1–1}$$

This force is directed along the line connecting the two points; G is Newton's gravitational constant. In SI units (Système International d'unités) based on meter [m], kilogram [kg], and second [s], the gravitational constant has the value

$$G = 6.6742 \cdot 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}\,. \tag{1–2}$$

The Newtonian gravitational constant G is somewhat of a scandal in measuring physics. It is on the one hand one of the most important physical constants, and at the same time one of the least accurately determined ones. The international authority in this field is the Committee on Data for Science and Technology (CODATA), see under www.codata.org. In July 2002, CODATA recommended the value of G mentioned above, more precisely it assigned the value $G = (6.6742 \pm 0.0010) \cdot 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. The symbol $\pm$ denotes the standard uncertainty, also called standard deviation or standard error. This corresponds to a relative standard uncertainty of $1.5 \cdot 10^{-4}$ or 150 ppm which is a deplorably high inaccuracy for such an important constant, see http://physics.nist.gov/cuu/constants. (For other constants we have a relative accuracy of 10^{-7} and better.) For comparison of experimental results see the internet.

Although the masses m_1, m_2 attract each other in a completely symmetrical way, it is convenient to call one of them the attracting mass and the other the attracted mass. For simplicity we set the attracted mass equal to unity and denote the attracting mass by m. The formula

$$F = G\,\frac{m}{l^2} \tag{1–3}$$

expresses the force exerted by the mass m on a unit mass located at P at a distance l from m.

We now introduce a rectangular coordinate system xyz and denote the coordinates of the attracting mass m by ξ, η, ζ and the coordinates of the attracted point P by x, y, z. The force may be represented by a vector $\mathbf{F}$ with magnitude F (Fig. 1.1). The components of $\mathbf{F}$ are given by

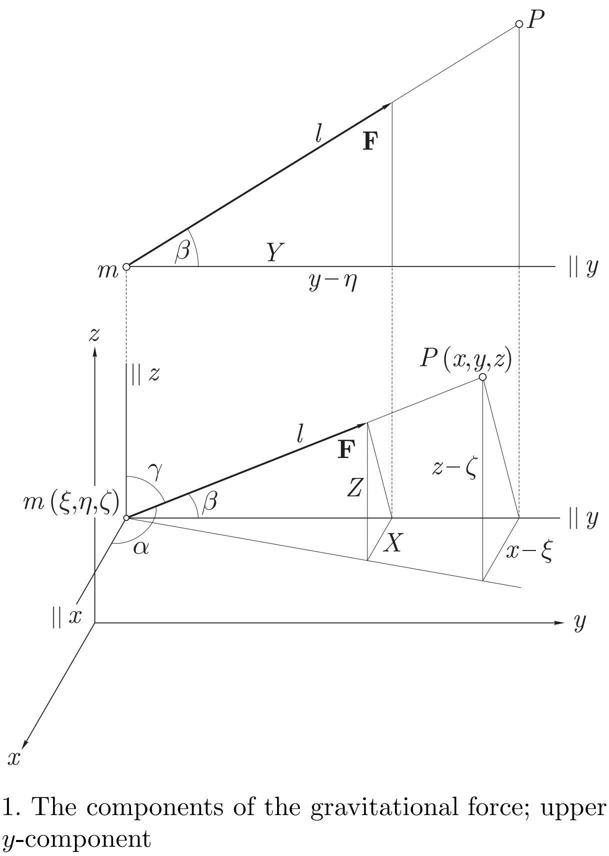

Fig. 1.1. The components of the gravitational force; upper figure shows y-component

$$
\begin{aligned}
X &= -F\cos\alpha = -\frac{G\,m}{l^2}\,\frac{x-\xi}{l} = -G\,m\,\frac{x-\xi}{l^3}\,,\\
Y &= -F\cos\beta = -\frac{G\,m}{l^2}\,\frac{y-\eta}{l} = -G\,m\,\frac{y-\eta}{l^3}\,,\\
Z &= -F\cos\gamma = -\frac{G\,m}{l^2}\,\frac{z-\zeta}{l} = -G\,m\,\frac{z-\zeta}{l^3}\,,
\end{aligned}
\tag{1–4}
$$

where

$$
l = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}\,. \tag{1–5}
$$

We next introduce a scalar function

$$
V = \frac{G\,m}{l}\,, \tag{1–6}
$$

called the potential of gravitation. The components X, Y, Z of the gravitational force $\mathbf{F}$ are then given by

$$
X = \frac{\partial V}{\partial x}\,,\quad Y = \frac{\partial V}{\partial y}\,,\quad Z = \frac{\partial V}{\partial z}\,, \tag{1–7}
$$

as can be easily verified by differentiating (1–6), since

$$
\frac{\partial}{\partial x}\left(\frac{1}{l}\right) = -\frac{1}{l^2}\,\frac{\partial l}{\partial x} = -\frac{1}{l^2}\,\frac{x-\xi}{l} = -\frac{x-\xi}{l^3}\,,\ \ldots\,. \tag{1–8}
$$

In vector notation, Eq. (1–7) is written

$$
\mathbf{F} = [X,\ Y,\ Z] = \operatorname{grad} V\,; \tag{1–9}
$$

that is, the force vector is the gradient vector of the scalar function V.

It is of basic importance that according to (1–7) the three components of the vector $\mathbf{F}$ can be replaced by a single function V. Especially when we consider the attraction of systems of point masses or of solid bodies, as we do in geodesy, it is much easier to deal with the potential than with the three components of the force. Even in these complicated cases the relations (1–7) are applied; the function V is then simply the sum of the contributions of the respective particles.

Thus, if we have a system of several point masses $m_1, m_2, \ldots, m_n$, the potential of the system is the sum of the individual contributions (1–6):

$$
V = \frac{G\,m_1}{l_1} + \frac{G\,m_2}{l_2} + \cdots + \frac{G\,m_n}{l_n} = G\sum_{i=1}^{n}\frac{m_i}{l_i}\,. \tag{1–10}
$$

1.2 Potential of a solid body

Let us now assume that point masses are distributed continuously over a volume v (Fig. 1.2) with density

$$\varrho = \frac{dm}{dv} \,, \tag{1–11}$$

where dv is an element of volume and dm is an element of mass. Then the sum (1–10) becomes an integral (Newton's integral),

$$V = G \iiint\limits_v \frac{dm}{l} = G \iiint\limits_v \frac{\varrho}{l}\, dv \,, \tag{1–12}$$

where l is the distance between the mass element $dm = \varrho\, dv$ and the attracted point P. Denoting the coordinates of the attracted point P by x, y, z and of the mass element m by ξ, η, ζ, we see that l is again given by (1–5), and we can write explicitly

$$V(x,y,z) = G \iiint\limits_v \frac{\varrho(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}\, d\xi\, d\eta\, d\zeta \,, \tag{1–13}$$

since the element of volume is expressed by

$$dv = d\xi\, d\eta\, d\zeta \,. \tag{1–14}$$

This is the reason for the triple integrals in (1–12).

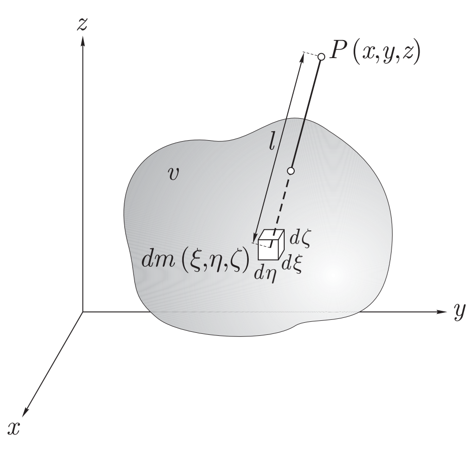

Fig. 1.2. Potential of a solid body

The components of the force of attraction are given by (1–7). For instance,

$$\begin{aligned} X = \frac{\partial V}{\partial x} &= G \frac{\partial}{\partial x} \iiint\limits_v \frac{\varrho(\xi,\eta,\zeta)}{l}\, d\xi\, d\eta\, d\zeta \\ &= G \iiint\limits_v \varrho(\xi,\eta,\zeta) \frac{\partial}{\partial x}\left(\frac{1}{l}\right) d\xi\, d\eta\, d\zeta \,. \end{aligned} \tag{1–15}$$

Note that we have interchanged the order of differentiation and integration. Substituting (1–8) into the above expression, we finally obtain

$$X = -G \iiint\limits_v \frac{x-\xi}{l^3}\, \varrho\, dv \,. \tag{1–16}$$

Analogous expressions result for Y and Z.

The potential V is continuous throughout the whole space and vanishes at infinity like $1/l$ for $l \to \infty$. This can be seen from the fact that for very large distances l the body acts approximately like a point mass, with the result that its attraction is then approximately given by (1–6). Consequently, in celestial mechanics the planets are usually considered as point masses.

The first derivatives of V, that is, the force components, are also continuous throughout space, but not so the second derivatives. At points where the density changes discontinuously, some second derivatives have a discontinuity. This is evident because the potential V may be shown to satisfy *Poisson's equation*

$$\Delta V = -4\pi\, G\, \varrho \,, \tag{1–17}$$

where

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \,. \tag{1–18}$$

The symbol Δ, called the *Laplacian operator*, has the form

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{1–19}$$

From (1–17) and (1–18) we see that at least one of the second derivatives of V must be discontinuous together with ϱ.

Outside the attracting bodies, in empty space, the density ϱ is zero and (1–17) becomes

$$\Delta V = 0 \,. \tag{1–20}$$

This is *Laplace's equation.* Its solutions are called *harmonic functions.* Hence, the potential of gravitation is a harmonic function outside the attracting masses but not inside the masses: there it satisfies Poisson's equation.

1.3 Harmonic functions

Earlier we have defined the harmonic functions as solutions of Laplace's equation

$$\Delta V = 0 \,. \tag{1–21}$$

More precisely, a function is called *harmonic in a region* v of space if it satisfies Laplace's equation at every point of v. If the region is the exterior of a certain closed surface S, then it must in addition vanish like $1/l$ for $l \to \infty$. It can be shown that *every harmonic function is analytic* (in the region where it satisfies Laplace's equation); that is, it is continuous and has continuous derivatives of any order and can be developed into a Taylor series.

The simplest harmonic function is the reciprocal distance

$$\frac{1}{l} = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \tag{1–22}$$

between two points $P(\xi, \eta, \zeta)$ and $P(x, y, z)$. It is the potential of a point mass $m = 1/G$, situated at the point $P(\xi, \eta, \zeta)$; compare (1–5) and (1–6).

It is easy to show that $1/l$ is harmonic. We form the following partial derivatives with respect to x, y, z in the fashion of (1–8):

$$\frac{\partial}{\partial x}\left(\frac{1}{l}\right) = -\frac{x-\xi}{l^3}\,, \quad \frac{\partial}{\partial y}\left(\frac{1}{l}\right) = -\frac{y-\eta}{l^3}\,, \quad \frac{\partial}{\partial z}\left(\frac{1}{l}\right) = -\frac{z-\zeta}{l^3}\,;$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{l}\right) = \frac{-l^2 + 3(x-\xi)^2}{l^3}\,, \quad \frac{\partial^2}{\partial y^2}\left(\frac{1}{l}\right) = \frac{-l^2 + 3(y-\eta)^2}{l^3}\,, \tag{1–23}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{l}\right) = \frac{-l^2 + 3(z-\zeta)^2}{l^3}\,.$$

Adding the last three equations and recalling the definition of Δ, we find

$$\Delta\left(\frac{1}{l}\right) = 0\,; \tag{1–24}$$

that is, $1/l$ is harmonic.

The point $P(\xi, \eta, \zeta)$, where l is zero and $1/l$ is infinite, is the only point to which we cannot apply the above derivation; $1/l$ is not harmonic at this singular point.

As a matter of fact, the slightly more general potential (1–6) of an arbitrary point mass m is also harmonic except at $P(\xi, \eta, \zeta)$, because (1–24) remains unchanged if both sides are multiplied by Gm.

Not only the potential of a point mass but also any other gravitational potential is harmonic outside the attracting masses. Consider the potential (1–12) of an extended body. Interchanging the order of differentiation and integration, we find from (1–12)

$$\Delta V = G\,\Delta\left[\iiint\limits_v \frac{\varrho}{l}\,dv\right] = G\iiint\limits_v \varrho\,\Delta\left(\frac{1}{l}\right)\,dv = 0\,; \tag{1–25}$$

that is, the potential of a solid body is also harmonic at any point $P(x,y,z)$ outside the attracting masses.

If P lies inside the attracting body, the above derivation breaks down, since $1/l$ becomes infinite for that mass element $dm(\xi,\eta,\zeta)$ which coincides with $P(x,y,z)$, and (1–24) does not apply. This is the reason why the potential of a solid body is not harmonic in its interior but instead satisfies Poisson's differential equation (1–17).

1.4 Laplace's equation in spherical coordinates

The most important harmonic functions are the *spherical harmonics*. To find them, we introduce spherical coordinates: r (radius vector; note that this is a standard notation, although it does not represent a vector in the contemporary sense), ϑ (polar distance), λ (geocentric longitude), see Fig. 1.3. Spherical coordinates are related to rectangular coordinates x, y, z by the

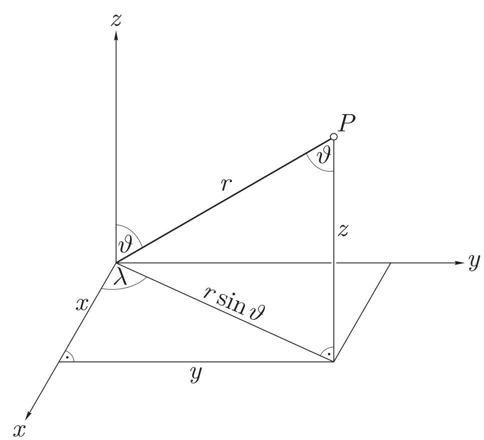

Fig. 1.3. Spherical and rectangular coordinates

equations

$$\begin{aligned} x &= r \sin\vartheta \cos\lambda\,, \\ y &= r \sin\vartheta \sin\lambda\,, \\ z &= r \cos\vartheta\,; \end{aligned} \tag{1–26}$$

or inversely by

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2}\,, \\ \vartheta &= \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}\,, \\ \lambda &= \tan^{-1} \frac{y}{x}\,. \end{aligned} \tag{1–27}$$

To get Laplace's equation in spherical coordinates, we first determine the element of arc (element of distance) ds in these coordinates. For this purpose we form

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \vartheta}\,d\vartheta + \frac{\partial x}{\partial \lambda}\,d\lambda\,, \\ dy &= \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \vartheta}\,d\vartheta + \frac{\partial y}{\partial \lambda}\,d\lambda\,, \\ dz &= \frac{\partial z}{\partial r}\,dr + \frac{\partial z}{\partial \vartheta}\,d\vartheta + \frac{\partial z}{\partial \lambda}\,d\lambda\,. \end{aligned} \tag{1–28}$$

By differentiating (1–26) and substituting it into the elementary formula

$$ds^2 = dx^2 + dy^2 + dz^2\,, \tag{1–29}$$

we obtain

$$ds^2 = dr^2 + r^2\,d\vartheta^2 + r^2 \sin^2\!\vartheta\;d\lambda^2\,. \tag{1–30}$$

We might have found this well-known formula more simply by geometrical considerations, but the approach used here is more general and can also be applied to ellipsoidal (harmonic) coordinates.

In (1–30) there are no terms with $dr\,d\vartheta$, $dr\,d\lambda$, and $d\vartheta\,d\lambda$. This expresses the evident fact that spherical coordinates are orthogonal: the spheres $r =$ constant, the cones $\vartheta =$ constant, and the planes $\lambda =$ constant intersect each other orthogonally.

The general form of the element of arc in arbitrary orthogonal coordinates q_1, q_2, q_3 is

$$ds^2 = h_1^2\,dq_1^2 + h_2^2\,dq_2^2 + h_3^2\,dq_3^2\,. \tag{1–31}$$

It can be shown that Laplace's operator in these coordinates is

$$\Delta V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial q_3}\right)\right]. \tag{1–32}$$

For spherical coordinates we have $q_1 = r$, $q_2 = \vartheta$, $q_3 = \lambda$. Comparison of (1–30) and (1–31) shows that

$$h_1 = 1\,, \quad h_2 = r\,, \quad h_3 = r \sin\vartheta\,. \tag{1–33}$$

Substituting these relations into (1–32) yields

$$\Delta V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial V}{\partial\vartheta}\right) + \frac{1}{r^2 \sin^2\!\vartheta}\frac{\partial^2 V}{\partial\lambda^2}\,. \tag{1–34}$$

Performing the differentiations we find

$$\Delta V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\vartheta^2} + \frac{\cot\vartheta}{r^2}\frac{\partial V}{\partial\vartheta} + \frac{1}{r^2 \sin^2\!\vartheta}\frac{\partial^2 V}{\partial\lambda^2} = 0\,, \tag{1–35}$$

which is *Laplace's equation in spherical coordinates*. An alternative expression is obtained when multiplying both sides by r^2:

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial\vartheta^2} + \cot\vartheta\frac{\partial V}{\partial\vartheta} + \frac{1}{\sin^2\!\vartheta}\frac{\partial^2 V}{\partial\lambda^2} = 0\,. \tag{1–36}$$

This form will be somewhat more convenient for our subsequent development.

1.5 Spherical harmonics

We attempt to solve Laplace's equation (1–35) or (1–36) by separating the variables r, ϑ, λ using the trial substitution

$$V(r, \vartheta, \lambda) = f(r)\, Y(\vartheta, \lambda)\,, \tag{1–37}$$

where f is a function of r only and Y is a function of ϑ and λ only. Performing this substitution in (1–36) and dividing by $f\,Y$, we get

$$\frac{1}{f}\,(r^2 f'' + 2r\, f') = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial\vartheta^2} + \cot\vartheta\frac{\partial Y}{\partial\vartheta} + \frac{1}{\sin^2\!\vartheta}\frac{\partial^2 Y}{\partial\lambda^2}\right), \tag{1–38}$$

where the primes denote differentiation with respect to the argument (r, in this case). Since the left-hand side depends only on r and the right-hand side only on ϑ and λ, both sides must be constant. We can therefore separate the equation into two equations:

$$r^2 f''(r) + 2r\, f'(r) - n(n+1)\, f(r) = 0 \tag{1–39}$$

and

$$\frac{\partial^2 Y}{\partial \vartheta^2} + \cot\vartheta\, \frac{\partial Y}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\, \frac{\partial^2 Y}{\partial \lambda^2} + n(n+1)\, Y = 0\,, \tag{1–40}$$

where we have denoted the constant by $n(n+1)$.

Solutions of (1–39) are given by the functions

$$f(r) = r^n \quad \text{and} \quad f(r) = r^{-(n+1)}\,; \tag{1–41}$$

this should be verified by substitution. Denoting the still unknown solutions of (1–40) by $Y_n(\vartheta, \lambda)$, we see that Laplace's equation (1–35) is solved by the functions

$$V = r^n\, Y_n(\vartheta, \lambda) \quad \text{and} \quad V = \frac{Y_n(\vartheta, \lambda)}{r^{n+1}}\,. \tag{1–42}$$

These functions are called *solid spherical harmonics*, whereas the functions $Y_n(\vartheta, \lambda)$ are known as (Laplace's) *surface spherical harmonics*. Both kinds are called *spherical harmonics*; the kind referred to can usually be judged from the context.

Note that n is not an arbitrary constant but must be an integer $0, 1, 2, \ldots$ as we will see later. If a differential equation is linear, and if we know several solutions, then, as is well known, the sum of these solutions is also a solution (this holds for *all* linear equation systems!). Hence, we conclude that

$$V = \sum_{n=0}^{\infty} r^n\, Y_n(\vartheta, \lambda) \quad \text{and} \quad V = \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \lambda)}{r^{n+1}} \tag{1–43}$$

are also solutions of Laplace's equation $\Delta V = 0$; that is, harmonic functions. The important fact is that *every* harmonic function – with certain restrictions – can be expressed in one of the forms (1–43).

1.6 Surface spherical harmonics

Now we have to determine Laplace's surface spherical harmonics $Y_n(\vartheta, \lambda)$. We attempt to solve (1–40) by a new trial substitution

$$Y_n(\vartheta, \lambda) = g(\vartheta)\; h(\lambda)\,, \tag{1–44}$$

where the functions g and h individually depend on one variable only. Performing this substitution in (1–40) and multiplying by $\sin^2\vartheta/g\,h$, we find

$$\frac{\sin\vartheta}{g}\left[\sin\vartheta\, g'' + \cos\vartheta\, g' + n(n+1)\sin\vartheta\, g\right] = -\frac{h''}{h}\,, \tag{1–45}$$

where the primes denote differentiation with respect to the argument: ϑ in g and λ in h. The left-hand side is a function of ϑ only, and the right-hand side is a function of λ only. Therefore, both sides must again be constant; let the constant be m^2. Thus, the partial differential equation (1–40) splits into two ordinary differential equations for the functions $g(\vartheta)$ and $h(\lambda)$:

$$\sin\vartheta\, g''(\vartheta) + \cos\vartheta\, g'(\vartheta) + \left[n(n+1)\sin\vartheta - \frac{m^2}{\sin\vartheta}\right] g(\vartheta) = 0\,; \tag{1–46}$$

$$h''(\lambda) + m^2 h(\lambda) = 0\,. \tag{1–47}$$

Solutions of Eq. (1–47) are the functions

$$h(\lambda) = \cos m\lambda \quad \text{and} \quad h(\lambda) = \sin m\lambda\,, \tag{1–48}$$

as may be verified by substitution. Equation (1–46), Legendre's differential equation, is more difficult. It can be shown that it has physically meaningful solutions only if n and m are integers $0, 1, 2, \ldots$ and if m is smaller than or equal to n. A solution of (1–46) is the Legendre function $P_{nm}(\cos\vartheta)$, which will be considered in some detail in the next section. Therefore,

$$g(\vartheta) = P_{nm}(\cos\vartheta) \tag{1–49}$$

and the functions

$$Y_n(\vartheta,\lambda) = P_{nm}(\cos\vartheta)\cos m\lambda \quad \text{and} \quad Y_n(\vartheta,\lambda) = P_{nm}(\cos\vartheta)\sin m\lambda \tag{1–50}$$

are solutions of the differential equation (1–40) for Laplace's surface spherical harmonics.

Since these solutions are linear, any linear combination of the solutions (1–50) is also a solution. Such a linear combination has the general form

$$Y_n(\vartheta,\lambda) = \sum_{m=0}^{n}\left[a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda + b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda\right], \tag{1–51}$$

where a_{nm} and b_{nm} are arbitrary constants. This is the general expression for the surface spherical harmonics $Y_n(\vartheta,\lambda)$.

Substituting this relation into equations (1–43), we see that

$$V_{\mathrm{i}}(r,\vartheta,\lambda)=\sum_{n=0}^{\infty} r^n \sum_{m=0}^{n}\left[a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda+b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda\right], \tag{1–52}$$

$$V_{\mathrm{e}}(r,\vartheta,\lambda)=\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}\left[a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda+b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda\right] \tag{1–53}$$

are solutions of Laplace's equation $\Delta V = 0$; that is, harmonic functions. Furthermore, as we have mentioned, they are very general solutions indeed: every function which is harmonic inside a certain sphere can be expanded into a series (1–52), where the subscript i indicates the interior, and every function which is harmonic outside a certain sphere (such as the earth's gravitational potential) can be expanded into a series (1–53), where the subscript e indicates the exterior. Thus, we see how spherical harmonics can be useful in geodesy.

1.7 Legendre's functions

In the preceding section we have introduced Legendre's function $P_{nm}(\cos\vartheta)$ as a solution of Legendre's differential equation (1–46). The subscript n denotes the *degree* and the subscript m the *order* of P_{nm}.

It is convenient to transform Legendre's differential equation (1–46) by the substitution

$$t=\cos\vartheta\,. \tag{1–54}$$

In order to avoid confusion, we use an overbar to denote g as a function of t. Therefore,

$$\begin{aligned} g(\vartheta) &= \bar{g}(t)\,,\\ g'(\vartheta) &= \frac{dg}{d\vartheta}=\frac{dg}{dt}\,\frac{dt}{d\vartheta}=-\bar{g}'(t)\sin\vartheta\,,\\ g''(\vartheta) &= \bar{g}''(t)\sin^2\!\vartheta-\bar{g}'(t)\cos\vartheta\,. \end{aligned} \tag{1–55}$$

Inserting these relations into (1–46), dividing by $\sin\vartheta$, and then substituting $\sin^2\!\vartheta = 1-t^2$, we get

$$(1-t^2)\,\bar{g}''(t)-2t\,\bar{g}'(t)+\left[n(n+1)-\frac{m^2}{1-t^2}\right]\bar{g}'(t)=0\,. \tag{1–56}$$

The Legendre function $\bar{g}(t)=P_{nm}(t)$, which is defined by

$$P_{nm}(t)=\frac{1}{2^n\,n!}(1-t^2)^{m/2}\,\frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n\,, \tag{1–57}$$

satisfies (1–56). Apart from the factor $(1-t^2)^{m/2} = \sin^m\vartheta$ and from a constant, the function P_{nm} is the $(n+m)$th derivative of the polynomial $(t^2-1)^n$. It can, thus, be evaluated. For instance,

$$P_{11}(t) = \frac{(1-t^2)^{1/2}}{2\cdot 1}\,\frac{d^2}{dt^2}(t^2-1) = \frac{1}{2}\sqrt{1-t^2}\cdot 2 = \sqrt{1-t^2} = \sin\vartheta\,. \quad (1\text{–}58)$$

The case $m=0$ is of particular importance. The functions $P_{n0}(t)$ are often simply denoted by $P_n(t)$. Then (1–57) gives

$$P_n(t) = P_{n0}(t) = \frac{1}{2^n\,n!}\,\frac{d^n}{dt^n}(t^2-1)^n\,. \quad (1\text{–}59)$$

Because $m=0$, there is no square root, that is, no $\sin\vartheta$. Therefore, the $P_n(t)$ are simply polynomials in t. They are called *Legendre's polynomials.* We give the Legendre polynomials for $n=0$ through $n=5$:

$$\begin{aligned} P_0(t) &= 1\,, & P_3(t) &= \tfrac{5}{2}\,t^3 - \tfrac{3}{2}\,t\,,\\ P_1(t) &= t\,, & P_4(t) &= \tfrac{35}{8}\,t^4 - \tfrac{15}{4}\,t^2 + \tfrac{3}{8}\,,\\ P_2(t) &= \tfrac{3}{2}\,t^2 - \tfrac{1}{2}\,, & P_5(t) &= \tfrac{63}{8}\,t^5 - \tfrac{35}{4}\,t^3 + \tfrac{15}{8}\,t\,. \end{aligned} \quad (1\text{–}60)$$

Remember that

$$t = \cos\vartheta\,. \quad (1\text{–}61)$$

The polynomials may be obtained by means of (1–59) or more simply by the *recursion formula*

$$P_n(t) = -\frac{n-1}{n}\,P_{n-2}(t) + \frac{2n-1}{n}\,t\,P_{n-1}(t)\,, \quad (1\text{–}62)$$

by which P_2 can be calculated from P_0 and P_1, P_3 from P_1 and P_2, etc. Graphs of the Legendre polynomials are shown in Fig. 1.4.

The powers of $\cos\vartheta$ can be expressed in terms of the cosines of multiples of ϑ, such as

$$\cos^2\vartheta = \tfrac{1}{2}\cos 2\vartheta + \tfrac{1}{2}\,, \quad \cos^3\vartheta = \tfrac{1}{4}\cos 3\vartheta + \tfrac{3}{4}\cos\vartheta\,. \quad (1\text{–}63)$$

Therefore, we may also express the $P_n(\cos\vartheta)$ in this way, obtaining

$$\begin{aligned} P_2(\cos\vartheta) &= \tfrac{3}{4}\cos 2\vartheta + \tfrac{1}{4}\,,\\ P_3(\cos\vartheta) &= \tfrac{5}{8}\cos 3\vartheta + \tfrac{3}{8}\cos\vartheta\,,\\ P_4(\cos\vartheta) &= \tfrac{35}{64}\cos 4\vartheta + \tfrac{5}{16}\cos 2\vartheta + \tfrac{9}{64}\,,\\ P_5(\cos\vartheta) &= \tfrac{63}{128}\cos 5\vartheta + \tfrac{35}{128}\cos 3\vartheta + \tfrac{15}{64}\cos\vartheta\,,\\ \cdots\quad &= \cdots\,. \end{aligned} \quad (1\text{–}64)$$

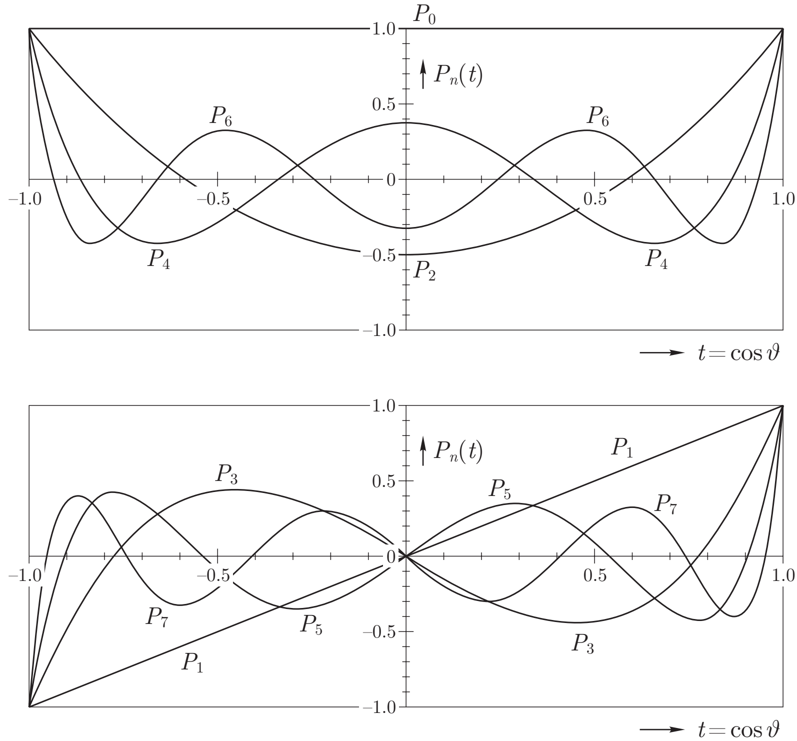

Fig. 1.4. Legendre's polynomials as functions of $t = \cos\vartheta$: n even (top) and n odd (bottom)

If the order m is not zero – that is, for $m = 1, 2, \ldots, n$ – Legendre's functions $P_{nm}(\cos\vartheta)$ are called *associated Legendre functions*. They can be reduced to the Legendre polynomials by means of the equation

$$P_{nm}(t) = (1 - t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}, \tag{1–65}$$

which follows from (1–57) and (1–59). Thus, the associated Legendre functions are expressed in terms of the Legendre polynomials of the same degree n. We give some P_{nm}, writing $t = \cos\vartheta$, $\sqrt{1 - t^2} = \sin\vartheta$:

$$\begin{aligned} P_{11}(\cos\vartheta) &= \sin\vartheta\,, & P_{31}(\cos\vartheta) &= \sin\vartheta\left(\tfrac{15}{2}\cos^2\vartheta - \tfrac{3}{2}\right),\\ P_{21}(\cos\vartheta) &= 3\sin\vartheta\cos\vartheta\,, & P_{32}(\cos\vartheta) &= 15\sin^2\vartheta\cos\vartheta\,,\\ P_{22}(\cos\vartheta) &= 3\sin^2\vartheta\,, & P_{33}(\cos\vartheta) &= 15\sin^3\vartheta\,. \end{aligned} \tag{1–66}$$

We also mention an explicit formula for any Legendre function (polynomial or associated function):

$$P_{nm}(t) = 2^{-n}(1-t^2)^{m/2} \sum_{k=0}^{r} (-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-m-2k)!}\, t^{n-m-2k}\,, \tag{1–67}$$

where r is the greatest integer $\leq (n-m)/2$; i.e., r is $(n-m)/2$ or $(n-m-1)/2$, whichever is an integer. This formula is convenient for programming.

As this useful formula is seldom found in the literature, we show the derivation, which is quite straightforward. The necessary information on factorials may be obtained from any collection of mathematical formulas. The binomial theorem gives

$$(t^2-1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^{2n-2k} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!}\, t^{2n-2k}\,. \tag{1–68}$$

Thus, (1–57) becomes

$$P_{nm}(t) = \frac{1}{2^n}(1-t^2)^{m/2} \sum_{k=0}^{n} (-1)^k \frac{1}{k!\,(n-k)!} \frac{d^{n+m}}{dt^{n+m}}(t^{2n-2k})\,, \tag{1–69}$$

the quantity $n!$ having been cancelled out. The rth derivative of the power t^s is

$$\frac{d^r}{dt^r}(t^s) = s(s-1)\cdots(s-r+1)\, t^{s-r} = \frac{s!}{(s-r)!}\, t^{s-r}\,. \tag{1–70}$$

Setting $r = n+m$ and $s = 2n-2k$, we have

$$\frac{d^{n+m}}{dt^{n+m}}(t^{2n-2k}) = \frac{(2n-2k)!}{(n-m-2k)!}\, t^{n-m-2k}\,. \tag{1–71}$$

Inserting this into the above expression for $P_{nm}(t)$ and noting that the lowest possible power of t is either t or $t^0 = 1$, we obtain (1–67).

The surface spherical harmonics are Legendre's functions multiplied by $\cos m\lambda$ or $\sin m\lambda$:

$$\begin{array}{ll}
\text{degree } 0 & P_0(\cos\vartheta)\,; \\[6pt]
\text{degree } 1 & P_1(\cos\vartheta)\,, \\
 & P_{11}(\cos\vartheta)\,\cos\lambda\,,\quad P_{11}(\cos\vartheta)\,\sin\lambda\,; \\[6pt]
\text{degree } 2 & P_2(\cos\vartheta)\,, \\
 & P_{21}(\cos\vartheta)\,\cos\lambda\,,\quad P_{21}(\cos\vartheta)\,\sin\lambda\,, \\
 & P_{22}(\cos\vartheta)\,\cos 2\lambda\,,\quad P_{22}(\cos\vartheta)\,\sin 2\lambda\,;
\end{array} \tag{1–72}$$

and so on.

The geometrical representation of these spherical harmonics is useful. The harmonics with $m = 0$ – that is, Legendre's polynomials – are polynomials of degree n in t, so that they have n zeros. These n zeros are all real and situated in the interval $-1 \leq t \leq +1$, that is, $0 \leq \vartheta \leq \pi$ (Fig. 1.4). Therefore, the harmonics with $m = 0$ change their sign n times in this interval; furthermore, they do not depend on λ. Their geometrical representation is therefore similar to Fig. 1.5 a. Since they divide the sphere into zones, they are also called *zonal harmonics*.

The associated Legendre functions change their sign $n - m$ times in the interval $0 \leq \vartheta \leq \pi$. The functions $\cos m\lambda$ and $\sin m\lambda$ have $2m$ zeros in the interval $0 \leq \lambda < 2\pi$, so that the geometrical representation of the harmonics for $m \neq 0$ is similar to that of Fig. 1.5 b. They divide the sphere into compartments in which they are alternately positive and negative, somewhat like a chess board, and are called *tesseral harmonics*. "Tessera" means a square or rectangle, or also a tile. In particular, for $n = m$, they degenerate into functions that divide the sphere into positive and negative sectors, in which case they are called *sectorial harmonics*, see Fig. 1.5 c.

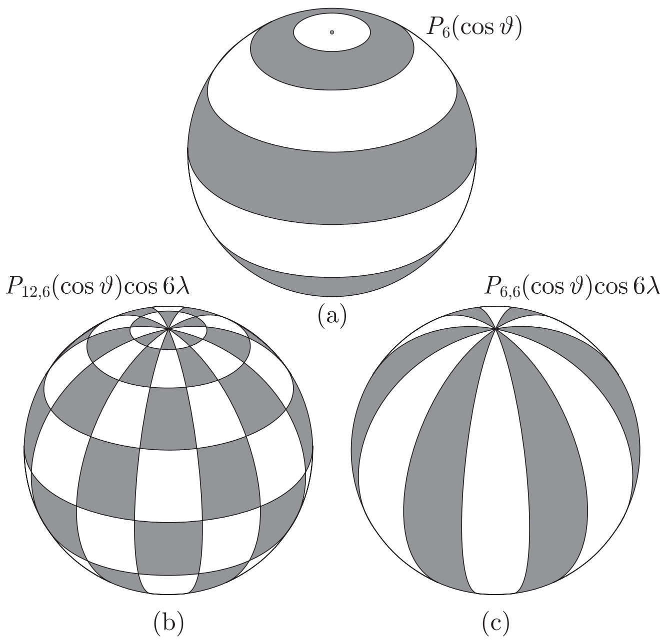

Fig. 1.5. The kinds of spherical harmonics: (a) zonal, (b) tesseral, (c) sectorial

1.8 Legendre's functions of the second kind

The Legendre function $P_{nm}(t)$ is not the only solution of Legendre's differential equation (1–56). There is a completely different function which also satisfies this equation. It is called *Legendre's function of the second kind*, of degree n and order m, and denoted by $Q_{nm}(t)$.

Although the $Q_{nm}(t)$ are functions of a completely different nature, they satisfy relationships very similar to those satisfied by the $P_{nm}(t)$.

The "zonal" functions

$$Q_n(t) \equiv Q_{n0}(t) \tag{1–73}$$

are defined by

$$Q_n(t) = \frac{1}{2} P_n(t) \ln \frac{1+t}{1-t} - \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(t) P_{n-k}(t) , \tag{1–74}$$

and the others by

$$Q_{nm}(t) = (1-t^2)^{m/2} \frac{d^m Q_n(t)}{dt^m} . \tag{1–75}$$

Equation (1–75) is completely analogous to (1–65); furthermore, the functions $Q_n(t)$ satisfy the same recursion formula (1–62) as the functions $P_n(t)$.

If we evaluate the first few Q_n, from (1–74) we find

$$\begin{aligned}
Q_0(t) &= \frac{1}{2} \ln \frac{1+t}{1-t} = \tanh^{-1} t , \\
Q_1(t) &= \frac{t}{2} \ln \frac{1+t}{1-t} - 1 = t \tanh^{-1} t - 1 , \\
Q_2(t) &= \left(\frac{3}{4} t^2 - \frac{1}{4} \right) \ln \frac{1+t}{1-t} - \frac{3}{2} t = \left(\frac{3}{2} t^2 - \frac{1}{2} \right) \tanh^{-1} t - \frac{3}{2} t .
\end{aligned} \tag{1–76}$$

These formulas and Fig. 1.6 show that the functions Q_{nm} are really quite different from the functions P_{nm}. From the singularity $\pm\infty$ at $t = \pm 1$ (i.e., $\vartheta = 0$ or π), we see that it is impossible to substitute $Q_{nm}(\cos\vartheta)$ for $P_{nm}(\cos\vartheta)$ if ϑ means the polar distance, because harmonic functions must be regular.

However, we will encounter them in the theory of ellipsoidal harmonics (Sect. 1.16), which is applied to the normal gravity field of the earth (Sect. 2.7). For this purpose we need Legendre's functions of the second

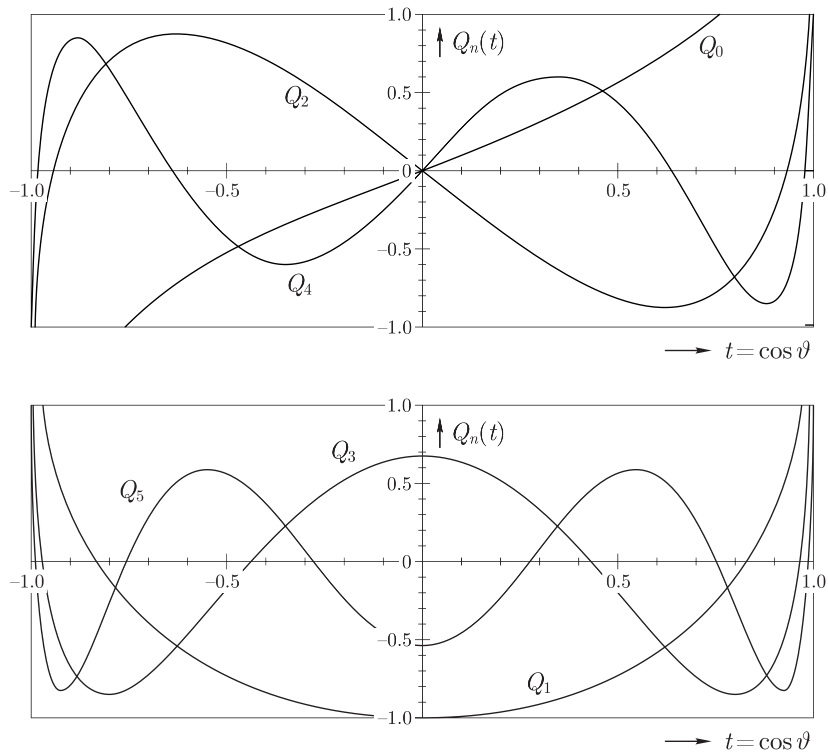

Fig. 1.6. Legendre's functions of the second kind: n even (top) and n odd (bottom)

kind as functions of a complex argument. If the argument z is complex, we must replace the definition (1–74) by

$$Q_n(z) = \frac{1}{2}\, P_n(z)\, \ln \frac{z+1}{z-1} - \sum_{k=1}^{n} \frac{1}{k}\, P_{k-1}(z)\, P_{n-k}(z)\,, \tag{1–77}$$

where Legendre's polynomials $P_n(z)$ are defined by the same formulas as in the case of a real argument t. Therefore, the only change as compared to (1–74) is the replacement of

$$\frac{1}{2}\, \ln \frac{1+t}{1-t} = \tanh^{-1} t \tag{1–78}$$

by

$$\frac{1}{2}\, \ln \frac{z+1}{z-1} = \coth^{-1} z\,. \tag{1–79}$$

In particular, we have

$$
\begin{aligned}
Q_0(z) &= \frac{1}{2}\ \ln\frac{z+1}{z-1} = \coth^{-1} z\,,\\
Q_1(z) &= \frac{z}{2}\ \ln\frac{z+1}{z-1} - 1 = z\,\coth^{-1} z - 1\,,\\
Q_2(z) &= \left(\frac{3}{4}\,z^2 - \frac{1}{4}\right)\ln\frac{z+1}{z-1} - \frac{3}{2}\,z = \left(\frac{3}{2}\,z^2 - \frac{1}{2}\right)\coth^{-1} z - \frac{3}{2}\,z\,.
\end{aligned}
\tag{1–80}
$$

1.9 Expansion theorem and orthogonality relations

In (1–52) and (1–53), we have expanded *harmonic* functions in space into series of *solid* spherical harmonics. In a similar way an *arbitrary* (at least in a very general sense) function $f(\vartheta, \lambda)$ on the surface of the sphere can be expanded into a series of *surface* spherical harmonics:

$$
f(\vartheta,\lambda) = \sum_{n=0}^{\infty} Y_n(\vartheta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[a_{nm}\mathcal{R}_{nm}(\vartheta,\lambda) + b_{nm}\mathcal{S}_{nm}(\vartheta,\lambda)\right], \tag{1–81}
$$

where we have introduced the abbreviations

$$
\begin{aligned}
\mathcal{R}_{nm}(\vartheta,\lambda) &= P_{nm}(\cos\vartheta)\ \cos m\lambda\,,\\
\mathcal{S}_{nm}(\vartheta,\lambda) &= P_{nm}(\cos\vartheta)\ \sin m\lambda\,.
\end{aligned}
\tag{1–82}
$$

The symbols a_{nm} and b_{nm} are constant coefficients, which we will now determine. Essential for this purpose are the *orthogonality relations.* These remarkable relations mean that the integral over the unit sphere of the product of any two *different* functions $\mathcal{R}_{nm}$ or $\mathcal{S}_{nm}$ is zero:

$$
\left.
\begin{aligned}
\iint_\sigma \mathcal{R}_{nm}(\vartheta,\lambda)\ \mathcal{R}_{sr}(\vartheta,\lambda)\, d\sigma &= 0\\
\iint_\sigma \mathcal{S}_{nm}(\vartheta,\lambda)\ \mathcal{S}_{sr}(\vartheta,\lambda)\, d\sigma &= 0
\end{aligned}
\right\}\quad \text{if } s \neq n \text{ or } r \neq m \text{ or both}\,;
$$

$$
\iint_\sigma \mathcal{R}_{nm}(\vartheta,\lambda)\ \mathcal{S}_{sr}(\vartheta,\lambda)\, d\sigma = 0 \qquad \text{in any case}\,. \tag{1–83}
$$

For the product of two *equal* functions $\mathcal{R}_{nm}$ or $\mathcal{S}_{nm}$, we have

$$
\begin{aligned}
&\iint_\sigma [\mathcal{R}_{n0}(\vartheta,\lambda)]^2\, d\sigma = \frac{4\pi}{2n+1}\,;\\
&\iint_\sigma [\mathcal{R}_{nm}(\vartheta,\lambda)]^2\, d\sigma = \iint_\sigma [\mathcal{S}_{nm}(\vartheta,\lambda)]^2\, d\sigma = \frac{2\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!} \qquad (m \neq 0)\,.
\end{aligned}
\tag{1–84}
$$

Note that there is no $\mathcal{S}_{n0}$, since $\sin 0\lambda = 0$. In these formulas we have used the abbreviation

$$\iint\limits_{\sigma} = \int_{\lambda=0}^{2\pi} \int_{\vartheta=0}^{\pi} \tag{1–85}$$

for the integral over the unit sphere. The expression

$$d\sigma = \sin\vartheta \, d\vartheta \, d\lambda \tag{1–86}$$

denotes the surface element of the unit sphere.

Now we turn to the determination of the coefficients a_{nm} and b_{nm} in (1–81). Multiplying both sides of the equation by a certain $\mathcal{R}_{sr}(\vartheta, \lambda)$ and integrating over the unit sphere gives

$$\iint\limits_{\sigma} f(\vartheta, \lambda) \, \mathcal{R}_{sr}(\vartheta, \lambda) \, d\sigma = a_{sr} \iint\limits_{\sigma} [\mathcal{R}_{sr}(\vartheta, \lambda)]^2 \, d\sigma \,, \tag{1–87}$$

since in the double integral on the right-hand side all terms except the one with $n = s$, $m = r$ will vanish according to the orthogonality relations (1–83). The integral on the right-hand side has the value given in (1–84), so that a_{sr} is determined. In a similar way we find b_{sr} by multiplying (1–81) by $S_{sr}(\vartheta, \lambda)$ and integrating over the unit sphere. The result is

$$a_{n0} = \frac{2n+1}{4\pi} \iint\limits_{\sigma} f(\vartheta, \lambda) \, P_n(\cos\vartheta) \, d\sigma \,;$$

$$\left.\begin{aligned} a_{nm} &= \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \iint\limits_{\sigma} f(\vartheta, \lambda) \, \mathcal{R}_{nm}(\vartheta, \lambda) \, d\sigma \\ b_{nm} &= \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \iint\limits_{\sigma} f(\vartheta, \lambda) \, \mathcal{S}_{nm}(\vartheta, \lambda) \, d\sigma \end{aligned}\right\} \quad (m \neq 0) \,. \tag{1–88}$$

The coefficients a_{nm} and b_{nm} can, thus, be determined by integration.

We note that the Laplace spherical harmonics $Y_n(\vartheta, \lambda)$ in (1–81) may also be found directly by the formula

$$Y_n(\vartheta, \lambda) = \frac{2n+1}{4\pi} \int_{\lambda'=0}^{2\pi} \int_{\vartheta'=0}^{\pi} f(\vartheta', \lambda') \, P_n(\cos\psi) \, \sin\vartheta' \, d\vartheta' \, d\lambda' \,, \tag{1–89}$$

where ψ is the spherical distance between the points P, represented by ϑ, λ, and P', represented by ϑ', λ' (Fig. 1.7), so that

$$\cos\psi = \cos\vartheta \cos\vartheta' + \sin\vartheta \sin\vartheta' \cos(\lambda' - \lambda) \,. \tag{1–90}$$

Later, when being acquainted with Sect. 1.11, Eq. (1–89) may be verified by straightforward computation, substituting $P_n(\cos\psi)$ from the decomposition formula (1–105).

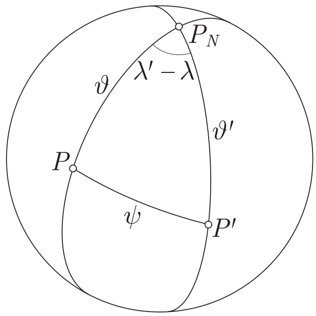

Fig. 1.7. Spherical distance ψ

1.10 Fully normalized spherical harmonics

The formulas of the preceding section for the expansion of a function into a series of surface spherical harmonics are somewhat inconvenient to handle. If we look at equations (1–84) and (1–88), we see that there are different formulas for $m = 0$ and $m \neq 0$; furthermore, the expressions are rather complicated and difficult to remember.

Therefore, it has been proposed that the "conventional" harmonics $\mathcal{R}_{nm}$ and $\mathcal{S}_{nm}$, defined by (1–82) together with (1–57), be replaced by other functions which differ by a constant factor and are easier to handle. We consider here only the *fully normalized* harmonics, which seem to be the most convenient and the most widely used.

The "fully normalized" harmonics are simply "normalized" in the sense of the theory of real functions; we have to use this clumsy expression because the term "normalized spherical harmonics" has already been used for other functions, unfortunately often for some that are not "normalized" at all in the mathematical sense.

We denote the fully normalized harmonics by $\bar{\mathcal{R}}_{nm}$ and $\bar{\mathcal{S}}_{nm}$; they are defined by

$$\bar{\mathcal{R}}_{n0}(\vartheta,\lambda) = \sqrt{2n+1}\,\mathcal{R}_{n0}(\vartheta,\lambda) \equiv \sqrt{2n+1}\,P_n(\cos\vartheta)\,;$$

$$\left.\begin{aligned} \bar{\mathcal{R}}_{nm}(\vartheta,\lambda) &= \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}}\,\mathcal{R}_{nm}(\vartheta,\lambda) \\ \bar{\mathcal{S}}_{nm}(\vartheta,\lambda) &= \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}}\,\mathcal{S}_{nm}(\vartheta,\lambda) \end{aligned}\right\}\quad (m \neq 0)\,. \tag{1–91}$$

The orthogonality relations (1–83) also apply for these fully normalized harmonics, whereas Eqs. (1–84) are thoroughly simplified: they become

$$\frac{1}{4\pi}\iint_\sigma \bar{\mathcal{R}}_{nm}^2\,d\sigma = \frac{1}{4\pi}\iint_\sigma \bar{\mathcal{S}}_{nm}^2\,d\sigma = 1\,. \tag{1–92}$$

This means that the *average square of any fully normalized harmonic is unity*, the average being taken over the sphere (the average corresponds to the integral divided by the area 4π). This formula now applies for any m, whether it is zero or not.

If we expand an arbitrary function $f(\vartheta,\lambda)$ into a series of fully normalized harmonics, analogously to (1–81),

$$f(\vartheta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left[\bar{a}_{nm}\bar{\mathcal{R}}_{nm}(\vartheta,\lambda) + \bar{b}_{nm}\bar{\mathcal{S}}_{nm}(\vartheta,\lambda)\right], \tag{1–93}$$

then the coefficients $\bar{a}_{nm}, \bar{b}_{nm}$ are simply given by

$$\begin{aligned}\bar{a}_{nm} &= \frac{1}{4\pi}\iint_\sigma f(\vartheta,\lambda)\,\bar{\mathcal{R}}_{nm}(\vartheta,\lambda)\,d\sigma\,,\\ \bar{b}_{nm} &= \frac{1}{4\pi}\iint_\sigma f(\vartheta,\lambda)\,\bar{\mathcal{S}}_{nm}(\vartheta,\lambda)\,d\sigma\,;\end{aligned} \tag{1–94}$$

that is, the coefficients are the average products of the function and the corresponding harmonic $\bar{\mathcal{R}}_{nm}$ or $\bar{\mathcal{S}}_{nm}$.

The simplicity of formulas (1–92) and (1–94) constitutes the main advantage of the fully normalized spherical harmonics and makes them useful in many respects, even though the functions $\bar{\mathcal{R}}_{nm}$ and $\bar{\mathcal{S}}_{nm}$ in (1–91) are a little more complicated than the conventional $\mathcal{R}_{nm}$ and $\mathcal{S}_{nm}$. We have

$$\begin{aligned}\bar{\mathcal{R}}_{nm}(\vartheta,\lambda) &= \bar{P}_{nm}(\cos\vartheta)\,\cos m\lambda\,,\\ \bar{\mathcal{S}}_{nm}(\vartheta,\lambda) &= \bar{P}_{nm}(\cos\vartheta)\,\sin m\lambda\,,\end{aligned} \tag{1–95}$$

where

$$\bar{P}_{n0}(t) = \sqrt{2n+1}\,2^{-n}\sum_{k=0}^{r}(-1)^k\,\frac{(2n-2k)!}{k!\,(n-k)!\,(n-2k)!}\,t^{n-2k} \tag{1–96}$$

for $m = 0$, and

$$\begin{aligned}\bar{P}_{nm}(t) &= \sqrt{2(2n+1)\,\frac{(n-m)!}{(n+m)!}}\;2^{-n}\,(1-t^2)^{m/2}\;\cdot\\ &\quad \sum_{k=0}^{r}(-1)^k\,\frac{(2n-2k)!}{k!\,(n-k)!\,(n-m-2k)!}\,t^{n-m-2k}\end{aligned} \tag{1–97}$$

for $m \neq 0$. This corresponds to (1–67); here, as in (1–67), r is the greatest integer $\leq (n-m)/2$.

There are relations between the coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ for fully normalized harmonics and the coefficients a_{nm} and b_{nm} for conventional harmonics that are inverse to those in (1–91):

$$\begin{aligned} \bar{a}_{n0} &= \frac{a_{n0}}{\sqrt{2n+1}}\,; \\ \left.\begin{aligned} \bar{a}_{nm} &= \sqrt{\frac{1}{2(2n+1)}\frac{(n+m)!}{(n-m)!}}\,a_{nm} \\ \bar{b}_{nm} &= \sqrt{\frac{1}{2(2n+1)}\frac{(n+m)!}{(n-m)!}}\,b_{nm} \end{aligned}\right\} &\quad (m \neq 0)\,. \end{aligned} \tag{1–98}$$

1.11 Expansion of the reciprocal distance into zonal harmonics and decomposition formula

The distance l between two points with spherical coordinates

$$P(r, \vartheta, \lambda), \quad P'(r', \vartheta', \lambda') \tag{1–99}$$

is given by

$$l^2 = r^2 + r'^2 - 2r\,r'\cos\psi\,, \tag{1–100}$$

where ψ is the angle between the radius vectors r and r' (Fig. 1.8), so that, from (1–90),

$$\cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\lambda' - \lambda) \tag{1–101}$$

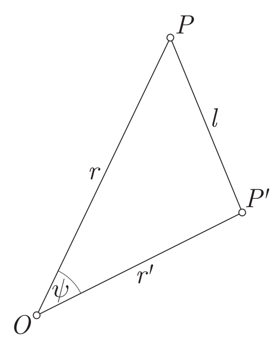

Fig. 1.8. The spatial distance l

results. Assuming $r' < r$, we may write

$$\frac{1}{l} = \frac{1}{\sqrt{r^2 - 2r\,r'\cos\psi + r'^2}} = \frac{1}{r\sqrt{1 - 2\alpha\,u + \alpha^2}}\,, \tag{1–102}$$

where we have put $\alpha = r'/r$ and $u = \cos\psi$. If $r' < r$, this can be expanded into a power series with respect to α. It is remarkable that the coefficients of α^n are the (conventional) zonal harmonics, or Legendre's polynomials $P_n(u) = P_n(\cos\psi)$:

$$\frac{1}{\sqrt{1 - 2\alpha\,u + \alpha^2}} = \sum_{n=0}^{\infty} \alpha^n\,P_n(u) = P_0(u) + \alpha\,P_1(u) + \alpha^2 P_2(u) + \cdots\,. \tag{1–103}$$

Hence, we obtain

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}}\,P_n(\cos\psi)\,, \tag{1–104}$$

which is an important formula.

It would still be desirable in this equation to express $P_n(\cos\psi)$ in terms of functions of the spherical coordinates ϑ, λ and ϑ', λ' of which ψ is composed according to (1–90). This is achieved by the *decomposition formula*

$$P_n(\cos\psi) = P_n(\cos\vartheta)\,P_n(\cos\vartheta') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!}\left[\mathcal{R}_{nm}(\vartheta,\lambda)\mathcal{R}_{nm}(\vartheta',\lambda') + \mathcal{S}_{nm}(\vartheta,\lambda)\mathcal{S}_{nm}(\vartheta',\lambda')\right]. \tag{1–105}$$

Substituting this into (1–104), we obtain

$$\frac{1}{l} = \sum_{n=0}^{\infty}\left\{\frac{P_n(\cos\vartheta)}{r^{n+1}}\,r'^n\,P_n(\cos\vartheta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}\cdot \left[\frac{\mathcal{R}_{nm}(\vartheta,\lambda)}{r^{n+1}}\,r'^n\,\mathcal{R}_{nm}(\vartheta',\lambda') + \frac{\mathcal{S}_{nm}(\vartheta,\lambda)}{r^{n+1}}\,r'^n\,\mathcal{S}_{nm}(\vartheta',\lambda')\right]\right\}. \tag{1–106}$$

The use of fully normalized harmonics simplifies these formulas. Replacing the conventional harmonics in (1–105) and (1–106) by fully normalized harmonics by means of (1–91), we find

$$P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=0}^{n}\left[\bar{\mathcal{R}}_{nm}(\vartheta,\lambda)\bar{\mathcal{R}}_{nm}(\vartheta',\lambda') + \bar{\mathcal{S}}_{nm}(\vartheta,\lambda)\bar{\mathcal{S}}_{nm}(\vartheta',\lambda')\right]; \tag{1–107}$$

$$\frac{1}{l} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{2n+1} \cdot \left[\frac{\bar{\mathcal{R}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \, r'^n \, \bar{\mathcal{R}}_{nm}(\vartheta', \lambda') + \frac{\bar{\mathcal{S}}_{nm}(\vartheta, \lambda)}{r^{n+1}} \, r'^n \, \bar{\mathcal{S}}_{nm}(\vartheta', \lambda') \right] . \tag{1–108}$$

The last formula will be fundamental for the expansion of the earth's gravitational field in spherical harmonics.

1.12 Solution of Dirichlet's problem by means of spherical harmonics and Poisson's integral

We define *Dirichlet's problem*, or the *first boundary-value problem of potential theory*, as follows: Given is an arbitrary function on a surface S, to determine is a function V which is harmonic either inside or outside S and which assumes on S the values of the prescribed function.

If the surface S is a sphere, then Dirichlet's problem can be solved by means of spherical harmonics. Let us take first the unit sphere, $r = 1$, and expand the prescribed function, given on the unit sphere and denoted by $V(1, \vartheta, \lambda)$, into a series of surface spherical harmonics (1–81):

$$V(1, \vartheta, \lambda) = \sum_{n=0}^{\infty} Y_n(\vartheta, \lambda) \,, \tag{1–109}$$

the $Y_n(\vartheta, \lambda)$ being determined by (1–89). (This series converges for very general functions V.) The functions

$$V_\mathrm{i}(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} r^n \, Y_n(\vartheta, \lambda) \tag{1–110}$$

and

$$V_\mathrm{e}(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} \frac{Y_n(\vartheta, \lambda)}{r^{n+1}} \tag{1–111}$$

assume the given values $V(1, \vartheta, \lambda)$ on the surface $r = 1$. The series (1–109) converges, and we have for $r < 1$

$$r^n \, Y_n < Y_n \tag{1–112}$$

and for $r > 1$

$$\frac{Y_n}{r^{n+1}} < Y_n \, . \tag{1–113}$$

Hence, the series (1–110) converges for $r \leq 1$, and the series (1–111) converges for $r \geq 1$; furthermore, both series have been found to represent harmonic functions. Therefore, we see that Dirichlet's problem is solved by $V_\mathrm{i}(r, \vartheta, \lambda)$ for the interior of the sphere $r = 1$, and by $V_\mathrm{e}(r, \vartheta, \lambda)$ for its exterior.

For a sphere of arbitrary radius $r = R$, the solution is similar. We expand the given function

$$V(R, \vartheta, \lambda) = \sum_{n=0}^{\infty} Y_n(\vartheta, \lambda) \,. \tag{1–114}$$

The surface spherical harmonics Y_n are determined by

$$Y_n(\vartheta, \lambda) = \frac{2n+1}{4\pi} \int_{\lambda'=0}^{2\pi} \int_{\vartheta'=0}^{\pi} V(R, \vartheta', \lambda')\, P_n(\cos\psi) \sin\vartheta' \, d\vartheta' \, d\lambda' \,. \tag{1–115}$$

Then the series

$$V_\mathrm{i}(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n Y_n(\vartheta, \lambda) \tag{1–116}$$

solves the first boundary-value problem for the interior, and the series

$$V_\mathrm{e}(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} Y_n(\vartheta, \lambda) \tag{1–117}$$

solves it for the exterior of the sphere $r = R$.

Thus, we see that Dirichlet's problem can always be solved for the sphere. It is evident that this is closely related to the possibility of expanding an *arbitrary* function on the sphere into a series of *surface* spherical harmonics and a *harmonic* function in space into a series of *solid* spherical harmonics.

Dirichlet's boundary-value problem can be solved not only for the sphere but also for any sufficiently smooth boundary surface. An example is given in Sect. 1.16.

The solvability of Dirichlet's problem is also essential to Molodensky's problem (Sect. 8.3). See also Kellogg (1929: Chap. XI).

Poisson's integral

A more direct solution is obtained as follows. We consider only the exterior problem, which is of greater interest in geodesy. Substituting $Y_n(\vartheta, \lambda)$ from (1–89) into (1–117), we obtain

$$V_\mathrm{e}(r, \vartheta, \lambda) =$$

$$\sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{2n+1}{4\pi} \int_{\lambda'=0}^{2\pi} \int_{\vartheta'=0}^{\pi} V(R, \vartheta', \lambda')\, P_n(\cos\psi) \sin\vartheta' \, d\vartheta' \, d\lambda' \,. \tag{1–118}$$

We can rearrange this as

$$V_e(r,\vartheta,\lambda) = \frac{1}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi} V(R,\vartheta',\lambda')\cdot \left[\sum_{n=0}^{\infty}(2n+1)\left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi)\right]\sin\vartheta'\,d\vartheta'\,d\lambda'\,. \tag{1–119}$$

The sum in the brackets can be evaluated. We denote the spatial distance between the points $P(r,\vartheta,\lambda)$ and $P'(R,\vartheta',\lambda')$ by l. Then, using (1–104),

$$\frac{1}{l} = \frac{1}{\sqrt{r^2+R^2-2Rr\cos\psi}} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi) \tag{1–120}$$

results. Differentiating with respect to r, we get

$$-\frac{r-R\cos\psi}{l^3} = -\frac{1}{R}\sum_{n=0}^{\infty}(n+1)\,\frac{R^{n+1}}{r^{n+2}}\,P_n(\cos\psi)\,. \tag{1–121}$$

Multiplying this equation by $-2Rr$, multiplying the expression for $1/l$ by $-R$, and then adding the two equations yields

$$\frac{R(r^2-R^2)}{l^3} = \sum_{n=0}^{\infty}(2n+1)\left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi)\,. \tag{1–122}$$

The right-hand side is the bracketed expression in (1–119). Substituting the left-hand side, we finally obtain

$$V_e(r,\vartheta,\lambda) = \frac{R(r^2-R^2)}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi}\frac{V(R,\vartheta',\lambda')}{l^3}\sin\vartheta'\,d\vartheta'\,d\lambda'\,, \tag{1–123}$$

where

$$l = \sqrt{r^2+R^2-2Rr\cos\psi}\,. \tag{1–124}$$

This is *Poisson's integral.* It is an explicit solution of Dirichlet's problem for the exterior of the sphere, which has many applications in physical geodesy.

1.13 Other boundary-value problems

There are other similar boundary-value problems. In *Neumann's problem*, or the *second boundary-value problem of potential theory*, the normal derivative $\partial V/\partial n$ is given on the surface S, instead of the function V itself. The normal derivative is the derivative along the outward-directed surface normal n to

S. In the *third boundary-value problem*, a linear combination of V and of its normal derivative

$$h\,V + k\,\frac{\partial V}{\partial n} \tag{1–125}$$

is given on S.

For the sphere, the solution of these boundary-value problems is also easily expressed in terms of spherical harmonics. We consider the exterior problems only, because these are of special interest to geodesy.

In *Neumann's problem*, we expand the given values of $\partial V/\partial n$ on the sphere $r = R$ into a series of surface spherical harmonics:

$$\left(\frac{\partial V}{\partial n}\right)_{r=R} = \sum_{n=0}^{\infty} Y_n(\vartheta, \lambda)\,. \tag{1–126}$$

The harmonic function which solves Neumann's problem for the exterior of the sphere is then

$$V_{\mathrm{e}}(r, \vartheta, \lambda) = -R \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{Y_n(\vartheta, \lambda)}{n+1}\,. \tag{1–127}$$

To verify it, we differentiate with respect to r, getting

$$\frac{\partial V_{\mathrm{e}}}{\partial r} = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+2} Y_n(\vartheta, \lambda)\,. \tag{1–128}$$

Since for the sphere the normal coincides with the radius vector, we have

$$\left(\frac{\partial V}{\partial n}\right)_{r=R} = \left(\frac{\partial V}{\partial r}\right)_{r=R}\,, \tag{1–129}$$

and we see that (1–126) is satisfied.

The *third boundary-value problem* is particularly relevant to physical geodesy, because the determination of the undulations of the geoid from gravity anomalies is just such a problem. To solve the general case, we again expand the function defined by the given boundary values into surface spherical harmonics:

$$h\,V + k\,\frac{\partial V}{\partial n} = \sum_{n=0}^{\infty} Y_n(\vartheta, \lambda)\,. \tag{1–130}$$

The harmonic function

$$V_{\mathrm{e}}(r, \vartheta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{Y_n(\vartheta, \lambda)}{h - (k/R)(n+1)} \tag{1–131}$$

solves the third boundary-value problem for the exterior of the sphere $r = R$. The straightforward verification is analogous to the case of (1–127).

In the determination of the geoidal undulations, the constants h, k have the values

$$h = -\frac{2}{R}, \qquad k = -1, \tag{1–132}$$

so that

$$V_\mathrm{e}(r, \vartheta, \lambda) = R \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \frac{Y_n(\vartheta, \lambda)}{n-1} \tag{1–133}$$

solves the *boundary-value problem of physical geodesy.*

As we have seen in the preceding section, the first boundary-value problem can also be solved directly by *Poisson's integral.* Similar integral formulas also exist for the second and the third problem. The integral formula that corresponds to (1–133) for the boundary-value problem of physical geodesy is *Stokes' integral*, which will be considered in detail in Chap. 2.

Remark on inverse problems

Boundary-value problems give the potential *outside* the earth, where there are no masses and where the potential, satisfying Laplace's equation, is harmonic. The determination of the potential *inside* the earth is of a quite different character since the earth is filled by masses, and the interior potential satisfies Poisson's rather than Laplace's equation, as we have seen in Sect. 1.2. Unfortunately, the density ϱ inside the earth is generally unknown.

To see the difficulties of the problem, let us consider Newton's integral (1–12). If the interior masses were known, we could easily use this formula to compute the potential inside (and outside) the earth, in a direct and straightforward way. The determination of the potential from the masses is a "direct" problem. The "inverse" problem is to determine the masses from the potential, finding a solution of Newton's integral for the density ϱ, which is essentially more difficult.

In fact, it is impossible to determine uniquely the generating masses from the external potential. This *inverse problem of potential theory* has no unique solution. Such inverse problems occur in geophysical prospecting by gravity measurements: underground masses are inferred from disturbances of the gravity field. To determine the problem more completely, additional information is necessary, which is furnished, for example, by geology or by seismic measurements.

Generally, nowadays we know that many problems in geophysics and other sciences including medicine (e.g., seismic and medical tomography) are inverse problems. We cannot pursue this interesting problem here and refer only to the extensive literature, e.g., the book by Moritz (1995), the internet page www.inas.tugraz.at/forschung/InverseProblems/AngerMoritz.html or Anger and Moritz (2003).

1.14 The radial derivative of a harmonic function

For later application to problems related with the vertical gradient of gravity, we will now derive an integral formula for the derivative along the radius vector r of an arbitrary harmonic function which we denote by V. Such a harmonic function satisfies Poisson's integral (1–123):

$$V(r,\vartheta,\lambda) = \frac{R(r^2-R^2)}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi}\frac{V(R,\vartheta',\lambda')}{l^3}\sin\vartheta'\,d\vartheta'\,d\lambda'\,. \tag{1–134}$$

Forming the radial derivative $\partial V/\partial r$, we note that $V(R,\vartheta',\lambda')$ does not depend on r. Thus, we need only to differentiate $(r^2-R^2)/l^3$, obtaining

$$\frac{\partial V(r,\vartheta,\lambda)}{\partial r} = \frac{R}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi} M(r,\psi)\,V(R,\vartheta',\lambda')\,\sin\vartheta'\,d\vartheta'\,d\lambda'\,, \tag{1–135}$$

where

$$M(r,\psi) \equiv \frac{\partial}{\partial r}\left(\frac{r^2-R^2}{l^3}\right) = \frac{1}{l^5}(5R^2r - r^3 - R\,r^2\cos\psi - 3R^3\cos\psi)\,. \tag{1–136}$$

Applying (1–135) to the special harmonic function $V_1(r,\vartheta,\lambda)=R/r$, where

$$\frac{\partial V_1}{\partial r} = -\frac{R}{r^2} \quad\text{and}\quad V_1(R,\vartheta',\lambda') = \frac{R}{R} = 1\,, \tag{1–137}$$

we obtain

$$-\frac{R}{r^2} = \frac{R}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi} M(r,\psi)\,\sin\vartheta'\,d\vartheta'\,d\lambda'\,. \tag{1–138}$$

Multiplying both sides of this equation by $V(r,\vartheta,\lambda)$ and subtracting it from (1–135) gives

$$\frac{\partial V}{\partial r} + \frac{R}{r^2}V_P = \frac{R}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\vartheta'=0}^{\pi} M(r,\psi)\,(V-V_P)\,\sin\vartheta'\,d\vartheta'\,d\lambda'\,, \tag{1–139}$$

where

$$V_P = V(r,\vartheta,\lambda)\,, \qquad V = V(R,\vartheta',\lambda')\,. \tag{1–140}$$

In order to find the radial derivative at the surface of the sphere of radius R, we must set $r=R$. Then l becomes (Fig. 1.9)

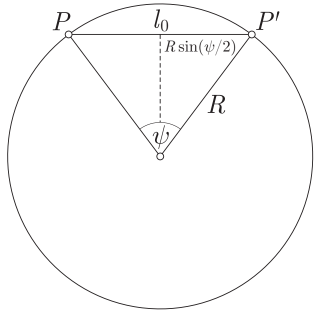

Fig. 1.9. Spatial distance between two points on a sphere

$$l_0 = 2R \sin \frac{\psi}{2} , \tag{1–141}$$

and the function M takes the simple form

$$M(R, \psi) = \frac{1}{4R^2 \sin^3 \frac{\psi}{2}} = \frac{2R}{l_0^3} . \tag{1–142}$$

For $\psi \to 0$ we have $M(R, \psi) \to \infty$, and we cannot use the original formula (1–135) at the surface of the sphere $r = R$. In the transformed equation (1–139), however, we have $V - V_P \to 0$ for $\psi \to 0$, and the singularity of M for $\psi \to 0$ will be neutralized (provided V is differentiable twice at P). Thus, we obtain the *gradient formula*

$$\frac{\partial V}{\partial r} = -\frac{1}{R} V_P + \frac{R^2}{2\pi} \int_{\lambda'=0}^{2\pi} \int_{\vartheta'=0}^{\pi} \frac{V - V_P}{l_0^3} \sin \vartheta' \, d\vartheta' \, d\lambda' . \tag{1–143}$$

This equation expresses $\partial V / \partial r$ on the sphere $r = R$ in terms of V on this sphere; thus, we now have

$$V_P = V(R, \vartheta, \lambda) , \qquad V = V(R, \vartheta', \lambda') . \tag{1–144}$$

Solution in terms of spherical harmonics

We may express V_P as

$$V_P = \sum_{n=0}^{\infty} \left(\frac{R}{r} \right)^{n+1} Y_n(\vartheta, \lambda) . \tag{1–145}$$

Differentiation yields

$$\frac{\partial V}{\partial r} = - \sum_{n=0}^{\infty} (n+1) \frac{R^{n+1}}{r^{n+2}} Y_n(\vartheta, \lambda) . \tag{1–146}$$

For $r = R$, this becomes

$$\frac{\partial V}{\partial r} = -\frac{1}{R} \sum_{n=0}^{\infty} (n+1) \, Y_n(\vartheta, \lambda) \, . \tag{1–147}$$

This is the equivalent of (1–143) in terms of spherical harmonics. From this equation, we get an interesting by-product. Writing (1–147) as

$$\frac{\partial V}{\partial r} = -\frac{1}{R} V_P - \frac{1}{R} \sum_{n=0}^{\infty} n \, Y_n(\vartheta, \lambda) \tag{1–148}$$

and comparing this with (1–143), we see that

$$\frac{R^2}{2\pi} \int_{\lambda'=0}^{2\pi} \int_{\vartheta'=0}^{\pi} \frac{V - V_P}{l_0^3} \sin \vartheta' \, d\vartheta' \, d\lambda' = -\frac{1}{R} \sum_{n=0}^{\infty} n \, Y_n(\vartheta, \lambda) \, . \tag{1–149}$$

This equation is formulated entirely in terms of quantities referred to the spherical surface only. Furthermore, for any function prescribed on the surface of a sphere, one can find a function in space that is harmonic outside the sphere and assumes the values of the function prescribed on it. This is done by solving Dirichlet's exterior problem. From these facts, we conclude that (1–149) holds for any (reasonably) arbitrary function V defined on the surface of a sphere. These developments will be used in Sect. 2.20.

1.15 Laplace's equation in ellipsoidal-harmonic coordinates

Spherical harmonics are most frequently used in geodesy because they are relatively simple and the earth is nearly spherical. Since the earth is more nearly an ellipsoid of revolution, it might be expected that ellipsoidal harmonics, which are defined in a way similar to that of the spherical harmonics, would be even more suitable. The whole matter is a question of mathematical convenience, since both spherical and ellipsoidal harmonics may be used for any attracting body, regardless of its form. As ellipsoidal harmonics are more complicated, however, they are used only in certain special cases which nevertheless are important, namely, in problems involving rigorous computation of normal gravity.

We introduce *ellipsoidal-harmonic coordinates* u, ϑ, λ (Fig. 1.10). In a rectangular system, a point P has the coordinates x, y, z. Now we pass through P the surface of an ellipsoid of revolution whose center is the origin O, whose rotation axis coincides with the z-axis, and whose linear eccentricity has the constant value E. The coordinate u is the semiminor axis of this

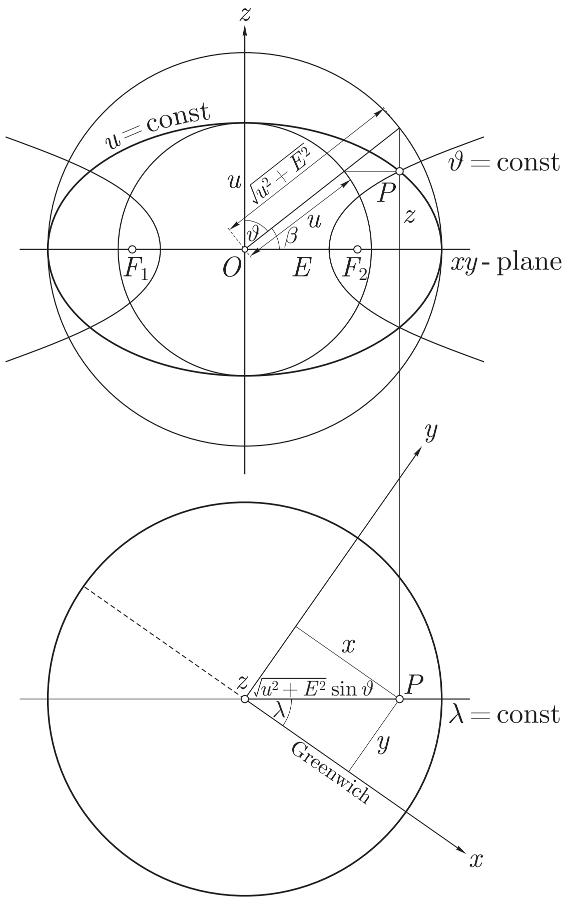

Fig. 1.10. Ellipsoidal-harmonic coordinates: view from the front (top) and view from above (bottom)

ellipsoid, ϑ is the complement of the "reduced latitude" β of P with respect to this ellipsoid (the definition is seen in Fig. 1.10), i.e., $\vartheta = 90° - \beta$, and λ is the geocentric longitude in the usual sense.

It should be carefully noted that in spherical harmonics ϑ is the polar distance, which is nothing but the complement of the *geocentric* latitude, whereas in ellipsoidal-harmonic coordinates ϑ is the complement of the *reduced* latitude denoted by β.

The ellipsoidal-harmonic coordinates u, ϑ, λ are related to x, y, z by

$$\begin{aligned} x &= \sqrt{u^2+E^2}\sin\vartheta\cos\lambda\,, \\ y &= \sqrt{u^2+E^2}\sin\vartheta\sin\lambda\,, \\ z &= u\cos\vartheta\,, \end{aligned} \tag{1–150}$$

which can be read from Fig. 1.10, considering that $\sqrt{u^2+E^2}$ is the semi-major axis of the ellipsoid whose surface passes through P. Because of $\vartheta = 90° - \beta$, we may equivalently write

$$\begin{aligned} x &= \sqrt{u^2+E^2}\cos\beta\cos\lambda\,,\\ y &= \sqrt{u^2+E^2}\cos\beta\sin\lambda\,,\\ z &= u\sin\beta\,. \end{aligned} \tag{1–151}$$

Taking $u =$ constant, we find

$$\frac{x^2+y^2}{u^2+E^2} + \frac{z^2}{u^2} = 1\,, \tag{1–152}$$

which represents an ellipsoid of revolution. For $\vartheta =$ constant, we obtain

$$\frac{x^2+y^2}{E^2\sin^2\vartheta} - \frac{z^2}{E^2\cos^2\vartheta} = 1\,, \tag{1–153}$$

which represents a hyperboloid of one sheet, and for $\lambda =$ constant, we get the meridian plane

$$y = x\tan\lambda\,. \tag{1–154}$$

The constant focal length E, i.e., the distance between the coordinate origin O and one of the focal points F_1 or F_2, which is the same for *all* ellipsoids $u =$ constant, characterizes the coordinate system. For $E = 0$ we have the usual spherical coordinates $u = r$ and ϑ, λ as a limiting case.

To find ds, the element of arc, in ellipsoidal-harmonic coordinates, we proceed in the same way as in spherical coordinates, Eq. (1–30), and obtain

$$ds^2 = \frac{u^2+E^2\cos^2\vartheta}{u^2+E^2}\,du^2 + (u^2+E^2\cos^2\vartheta)\,d\vartheta^2 + (u^2+E^2)\sin^2\vartheta\,d\lambda^2\,. \tag{1–155}$$

The coordinate system u, ϑ, λ is again orthogonal: the products $du\,d\vartheta$, etc., are missing in the equation above. Setting $u = q_1,\ \vartheta = q_2,\ \lambda = q_3$, we have in (1–31)

$$h_1^2 = \frac{u^2+E^2\cos^2\vartheta}{u^2+E^2}\,,\qquad h_2^2 = u^2+E^2\cos^2\vartheta\,,\qquad h_3^2 = (u^2+E^2)\sin^2\vartheta\,. \tag{1–156}$$

If we substitute these relations into (1–32), we obtain

$$\begin{aligned} \Delta V = &\frac{1}{(u^2+E^2\cos^2\vartheta)\sin\vartheta}\left\{\frac{\partial}{\partial u}\left[(u^2+E^2)\sin\vartheta\,\frac{\partial V}{\partial u}\right] + \right.\\ &\left.\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial V}{\partial\vartheta}\right) + \frac{\partial}{\partial\lambda}\left[\frac{u^2+E^2\cos^2\vartheta}{(u^2+E^2)\sin\vartheta}\,\frac{\partial V}{\partial\lambda}\right]\right\}\,. \end{aligned} \tag{1–157}$$

Performing the differentiations and cancelling $\sin\vartheta$, we get

$$\Delta V \equiv \frac{1}{u^2+E^2\cos^2\vartheta}\left[(u^2+E^2)\frac{\partial^2 V}{\partial u^2}+2u\frac{\partial V}{\partial u}+\frac{\partial^2 V}{\partial \vartheta^2}+\right.$$
$$\left.\cot\vartheta\frac{\partial V}{\partial\vartheta}+\frac{u^2+E^2\cos^2\vartheta}{(u^2+E^2)\sin^2\vartheta}\frac{\partial^2 V}{\partial\lambda^2}\right]=0\,, \tag{1–158}$$

which is *Laplace's equation in ellipsoidal-harmonic coordinates*. An alternative expression is obtained by omitting the factor $(u^2+E^2\cos^2\vartheta)^{-1}$:

$$(u^2+E^2)\frac{\partial^2 V}{\partial u^2}+2u\frac{\partial V}{\partial u}+\frac{\partial^2 V}{\partial \vartheta^2}+\cot\vartheta\frac{\partial V}{\partial\vartheta}+\frac{u^2+E^2\cos^2\vartheta}{(u^2+E^2)\sin^2\vartheta}\frac{\partial^2 V}{\partial\lambda^2}=0\,. \tag{1–159}$$

In the limiting case, $E\to 0$, these equations reduce to the spherical expressions (1–35) and (1–36).

1.16 Ellipsoidal harmonics

To solve (1–158) or (1–159), we proceed in a way which is analogous to the method used to solve the corresponding equation (1–36) in spherical coordinates. What we did there may be summarized as follows. By the trial substitution

$$V(r,\vartheta,\lambda)=f(r)\,g(\vartheta)\,h(\lambda)\,, \tag{1–160}$$

we separated the variables r,ϑ,λ, so that the original partial differential equation (1–36) was decomposed into three ordinary differential equations (1–39), (1–46), and (1–47).

In order to solve Laplace's equation in ellipsoidal coordinates (1–159), we correspondingly make the ansatz (trial substitution)

$$V(u,\vartheta,\lambda)=f(u)\,g(\vartheta)\,h(\lambda)\,. \tag{1–161}$$

Substituting and dividing by $f\,g\,h$, we get

$$\frac{1}{f}[(u^2+E^2)\,f''+2u\,f']+\frac{1}{g}(g''+g'\cot\vartheta)+\frac{u^2+E^2\cos^2\vartheta}{(u^2+E^2)\sin^2\vartheta}\,\frac{h''}{h}=0\,. \tag{1–162}$$

The variable λ occurs only through the quotient h''/h, which consequently must be constant. One sees this more clearly by writing the equation in the form

$$-\frac{(u^2+E^2)\sin^2\vartheta}{u^2+E^2\cos^2\vartheta}\left\{\frac{1}{f}[(u^2+E^2)\,f''+2u\,f']+\frac{1}{g}(g''+g'\cot\vartheta)\right\}=\frac{h''}{h}\,. \tag{1–163}$$

The left-hand side depends only on u and ϑ, the right-hand side only on λ. The two sides cannot be identically equal unless both are equal to the same constant. Therefore,

$$\frac{h''}{h} = -m^2 \,. \tag{1–164}$$

The factor by which h''/h is to be multiplied, i.e., the inverse of the main factor on the left-hand side of (1–163), can be decomposed as follows:

$$\frac{u^2 + E^2 \cos^2\vartheta}{(u^2 + E^2)\sin^2\vartheta} = \frac{1}{\sin^2\vartheta} - \frac{E^2}{u^2 + E^2} \,. \tag{1–165}$$

Substituting (1–164) and (1–165) into (1–163) and combining functions of the same variable, we obtain

$$\frac{1}{f}[(u^2 + E^2)\, f'' + 2u\, f'] + \frac{E^2}{u^2 + E^2}\, m^2 = -\frac{1}{g}(g'' + g' \cot\vartheta) + \frac{m^2}{\sin^2\vartheta} \,. \tag{1–166}$$

The two sides are functions of different independent variables and must therefore be constant. Denoting this constant by $n(n+1)$, we finally get

$$(u^2 + E^2)\, f''(u) + 2u\, f'(u) - \left[n(n+1) - \frac{E^2}{u^2 + E^2}\, m^2\right] f(u) = 0\,; \tag{1–167}$$

$$\sin\vartheta\, g''(\vartheta) + \cos\vartheta\, g'(\vartheta) + \left[n(n+1)\sin\vartheta - \frac{m^2}{\sin\vartheta}\right] g(\vartheta) = 0\,; \tag{1–168}$$

$$h''(\lambda) + m^2 h(\lambda) = 0\,. \tag{1–169}$$

These are the three ordinary differential equations into which the partial differential equation (1–159) is decomposed by the separation of variables (1–161).

The second and third equations are the same as in the spherical case, Eqs. (1–46) and (1–47); the first equation is different. The substitutions

$$\tau = i\,\frac{u}{E} \quad (\text{where } i = \sqrt{-1}) \quad \text{and} \quad t = \cos\vartheta \tag{1–170}$$

transform the first and second equations into

$$\begin{aligned} &(1 - \tau^2)\, \bar{f}''(\tau) - 2\tau\, \bar{f}'(\tau) + \left[n(n+1) - \frac{m^2}{1 - \tau^2}\right] \bar{f}(\tau) = 0\,, \\ &(1 - t^2)\, \bar{g}''(t) - 2t\, \bar{g}'(t) + \left[n(n+1) - \frac{m^2}{1 - t^2}\right] \bar{g}(t) = 0\,, \end{aligned} \tag{1–171}$$

where the overbar indicates that the functions f and g are expressed in terms of the new arguments τ and t. From spherical harmonics we are already familiar with the substitution $t = \cos\vartheta$ and the corresponding equation for $\bar{g}(t)$.

Note that $\bar{f}(\tau)$ satisfies formally the same differential equation as $\bar{g}(t)$, namely, Legendre's equation (1–56). As we have seen, this differential equation has two solutions: Legendre's function P_{nm} and Legendre's function of the second kind Q_{nm}. For $\bar{g}(t)$, where $t = \cos\vartheta$, the $Q_{nm}(t)$ are ruled out for obvious reasons, as we have seen in Sect. 1.8. For $\bar{f}(\tau)$, however, both sets of functions $P_{nm}(\tau)$ and $Q_{nm}(\tau)$ are possible solutions; they correspond to the two different solutions $f = r^n$ and $f = r^{-(n+1)}$ in the spherical case. Finally, (1–169) has as before the solutions $\cos m\lambda$ and $\sin m\lambda$.

We summarize all individual solutions:

$$\begin{aligned} f(u) &= P_{nm}\left(i\,\frac{u}{E}\right) \quad \text{or} \quad Q_{nm}\left(i\,\frac{u}{E}\right) \,; \\ g(\vartheta) &= P_{nm}(\cos\vartheta)\,; \\ h(\lambda) &= \cos m\lambda \quad \text{or} \quad \sin m\lambda\,. \end{aligned} \tag{1–172}$$

Here n and $m < n$ are integers $0, 1, 2, \ldots$, as before. Hence, the functions

$$\begin{aligned} V(u,\vartheta,\lambda) &= P_{nm}\left(i\,\frac{u}{E}\right) P_{nm}(\cos\vartheta) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} , \\ V(u,\vartheta,\lambda) &= Q_{nm}\left(i\,\frac{u}{E}\right) P_{nm}(\cos\vartheta) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \end{aligned} \tag{1–173}$$

are solutions of Laplace's equation $\Delta V = 0$, that is, harmonic functions.

From these functions we may form by linear combination the series

$$\begin{aligned} V_{\mathrm{i}}(u,\vartheta,\lambda) \;&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{P_{nm}\left(i\,\dfrac{u}{E}\right)}{P_{nm}\left(i\,\dfrac{b}{E}\right)} \cdot \\ &\quad [a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda + b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda]\,; \\ V_{\mathrm{e}}(u,\vartheta,\lambda) \;&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{Q_{nm}\left(i\,\dfrac{u}{E}\right)}{Q_{nm}\left(i\,\dfrac{b}{E}\right)} \cdot \\ &\quad [a_{nm}P_{nm}(\cos\vartheta)\cos m\lambda + b_{nm}P_{nm}(\cos\vartheta)\sin m\lambda]\,. \end{aligned} \tag{1–174}$$

Here b is the semiminor axis of an arbitrary but fixed ellipsoid which may be called the *reference ellipsoid* (Fig. 1.11). The division by $P_{nm}(ib/E)$ or $Q_{nm}(ib/E)$ is possible because they are constants; its purpose is to simplify the expressions and to make the coefficients a_{nm} and b_{nm} real.

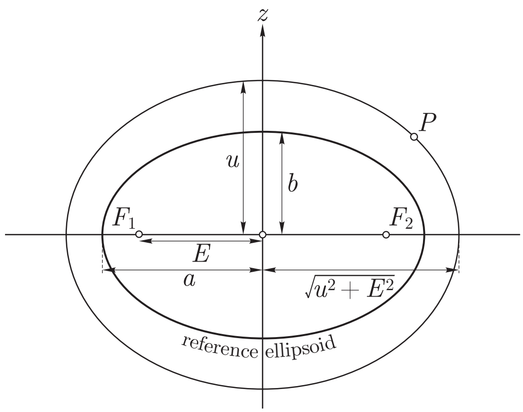

Fig. 1.11. Reference ellipsoid and ellipsoidal-harmonic coordinates

If the eccentricity E reduces to zero, the ellipsoidal-harmonic coordinates u, ϑ, λ become spherical coordinates r, ϑ, λ; the ellipsoid $u = b$ becomes the sphere $r = R$ because then the semiaxes a and b are equal to the radius R; and we find

$$\lim_{E\to 0} \frac{P_{nm}\left(i\,\dfrac{u}{E}\right)}{P_{nm}\left(i\,\dfrac{b}{E}\right)} = \left(\frac{u}{b}\right)^n = \left(\frac{r}{R}\right)^n, \quad \lim_{E\to 0} \frac{Q_{nm}\left(i\,\dfrac{u}{E}\right)}{Q_{nm}\left(i\,\dfrac{b}{E}\right)} = \left(\frac{R}{r}\right)^{n+1}, \tag{1–175}$$

so that the first series in (1–174) becomes (1–116), and the second series in (1–174) becomes (1–117). Thus, we see that the function $P_{nm}(iu/E)$ corresponds to r^n and $Q_{nm}(iu/E)$ corresponds to $r^{-(n+1)}$ in spherical harmonics.

Hence, the first series in (1–174) is harmonic in the interior of the ellipsoid $u = b$, and the second series in (1–174) is harmonic in its exterior; this case is relevant to geodesy. For $u = b$, the two series are equal:

$$\begin{aligned} V_\mathrm{i}(b, \vartheta, \lambda) &= V_\mathrm{e}(b, \vartheta, \lambda) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[a_{nm} P_{nm}(\cos\vartheta) \cos m\lambda + b_{nm} P_{nm}(\cos\vartheta) \sin m\lambda\right]. \end{aligned} \tag{1–176}$$

Thus, the solution of Dirichlet's boundary-value problem for the ellipsoid of revolution is easy. We expand the function $V(b, \vartheta, \lambda)$, given on the ellipsoid $u = b$, into a series of surface spherical harmonics with the following arguments: ϑ = complement of reduced latitude, λ = geocentric longitude. Then the first series in (1–174) is the solution of the interior problem and the second series in (1–174) is the solution of the exterior Dirichlet problem.

Formula (1–176) shows that not only functions that are defined on the surface of a sphere can be expanded into a series of surface spherical harmonics. Such an expansion is even possible for rather arbitrary functions defined on a convex surface.

A remark on terminology

The ellipsoidal-harmonic coordinates u, ϑ (or β), λ are the generalization of spherical coordinates for the sole use of getting closed solutions of Laplace's equation, in particular, for the gravity field of the reference ellipsoid in Sect. 2.7. The brief name "ellipsoidal coordinates" frequently used for u, β, λ might lead to a confusion with the ellipsoidal coordinates φ, λ, h. In the present book, "ellipsoidal coordinates" will always denote "ellipsoidal geographic coordinates", frequently also called "geodetic coordinates", being represented by φ, λ, h.