4 Beyond Circles and Ellipses

THE first three chapters discussed orbits and the work of space mission analysts. However, there are a variety of complex space missions these days that use trajectories that do not conform to the predominantly circular or elliptical orbits that we have discussed so far. Some of these, such as hyperbolic swing-bys, could be described as *ideal trajectories* in the sense that they remain under the umbrella of Isaac Newton's theory. However, recalling the hyperbolas briefly discussed in Chapter 1, these trajectories represent open trajectories that in no way resemble circular or elliptic orbits. In addition, some spacecraft operate in places where the gravitational (and other) forces acting upon them do not approximate to Newton's inverse square law. In these cases the spacecraft's orbit may not resemble a circle or an ellipse at all, and can be described as completely *non-Keplerian*.

In this chapter, we look briefly at some of these unusual trajectories that are becoming more popularly used by spacecraft operators. In particular we look at swing-by maneuvers past planets, orbits around small irregularly shaped bodies such as asteroids or comets, and halo orbits around Lagrangian points.

Swing-By Trajectories

If a spacecraft passes close by a planet, as it journeys through interplanetary space, then the path it takes is described as a *swing-by trajectory*. It may be worth recalling some of the background we discussed in Chapter 1 about this type of trajectory, and indeed you may wish to reread the text associated with Figures 1.9 and 1.10 to refresh your memory. The shape of the curve describing the spacecraft's path is called a hyperbola, and it is one of the four basic *conic section* shapes found by Isaac Newton in his equations describing motion in an *inverse square law gravity field*. As mentioned in Chapter 1, the shape can be seen all over the place once you start looking for it. You may even have your very own hyperbola in the room where you are reading this,

G. Swinerd, *How Spacecraft Fly: Spaceflight Without Formulae,*
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if you have a table or standard lamp placed adjacent to a wall (see Figure 1.10).

The shape of the hyperbolic swing-by trajectory is shown in Figure 4.1. When the spacecraft is a long way away from the planet, before it even gets to point A in the diagram, the gravity force of the planet is so feeble that the spacecraft effectively travels in a straight line. This line is called an *asymptote* of the hyperbola. However, as the spacecraft closes in on the planet, the gravity force steadily increases, and the spacecraft's path describes the classic hyperbolic shape depicted in Figure 4.1 from point A through point B to point C. Beyond point C, the gravity force decreases rapidly, and the trajectory tends to the straight line given by the asymptote once again. During this process, the planet has changed the direction of travel of the spacecraft, and this change is given by the *deflection angle,* which is effectively the angle between the incoming and outgoing asymptotes, as shown. The amount by which the trajectory is deflected is

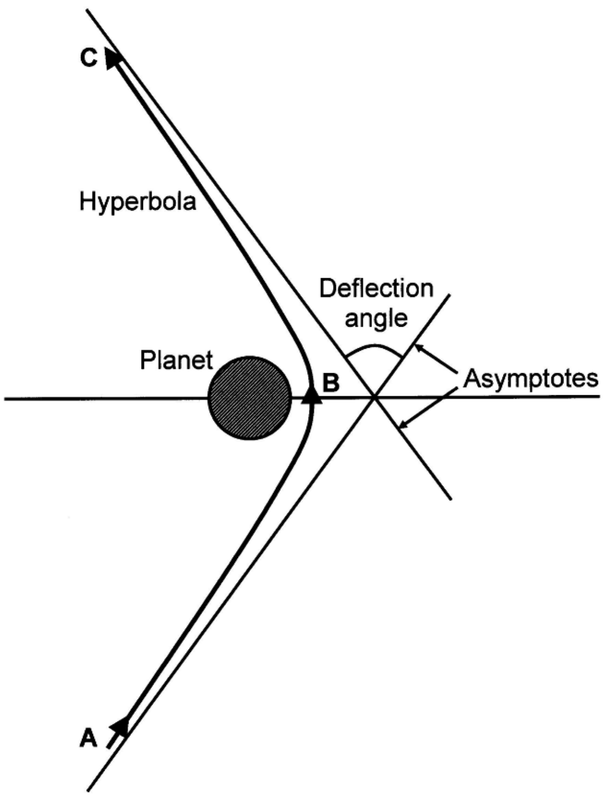

Figure 4.1: The classic hyperbolic shape of a swing-by maneuver past a planet.

dependent on three things: how massive the planet is, how close point B is to the planet, and how fast the spacecraft is traveling on approach.

This description echoes much of what was said in Chapter 1, but one thing that was not discussed was the speed of the spacecraft on the hyperbola. As the spacecraft falls toward the planet on the hyperbola, the gravity force increases, causing a corresponding increase in the vehicle's speed. Conversely, after the point of closest approach, the speed decreases again as the spacecraft climbs away against the force of gravity. The important thing to remember for what follows is that as speed and height are traded, and no energy is lost in the encounter, the speed of approach at A is identical to the speed of departure at C, relative to the planet. This is analogous to a cyclist negotiating a road that falls into a valley between two hilltops. The cyclist leaves the first hill at, say, 10 mph, and accelerates on the downhill section, reaching a maximum speed at the bottom of the valley. The cycle then slows down steadily on the uphill gradient on the other side of the valley, finally reaching the second hilltop at 10 mph again. To make the comparison with the spacecraft's motion complete, we have to assume that no energy is lost by the cyclist due to things like friction with the road or wind resistance. But nevertheless it is quite a good way of thinking about how the speed of the spacecraft varies on the hyperbola.

Gravity Assists

However, this is not the end of the story of hyperbolic swing-bys, because swing-bys can be used to increase the speed of spacecraft in their travels around the Sun without firing a rocket motor. (Swing-bys can also be used to decrease spacecraft speed, but we will not discuss this aspect.) Mission analysts get excited at the prospect of gaining spacecraft energy without having to use rocket fuel! It means that their spacecraft can achieve the journey to a distant planet more quickly, and do it without having to exchange payload mass for propellant mass. This type of maneuver is often referred to as a gravity assist, and it has been used many times by mission designers.

Perhaps the most famous example of its use is in the Voyager spacecraft program to explore the outer solar system. If we look back on the history of astronautics in the 20th century, the events that stand out for me are Sputnik 1, the first manned orbital flight by Yuri Gagarin, the Apollo moon landings, and the exploration of the outer solar system by the Voyager 1 and 2 spacecraft. Voyager 2 was launched in 1977, and took advantage of a rare alignment of the planets to visit Jupiter, Saturn, Uranus, and Neptune before finally leaving the solar system. The scientific return was huge! And it was achievable through the use of gravity-assist maneuvers at each planetary encounter, which increased the spacecraft's speed relative to the Sun so that the Neptune fly-by could be achieved after just 12 years from launch. Without this technique, and using the same launch energy, the transfer to Neptune would have taken around 30 years, and we would only now be learning about Neptune's mysterious moon Triton as I write this in 2006 (assuming that the spacecraft would still be operating after 30 years!).

Gravity-assist maneuvers clearly have great benefit, but how exactly do they work? Let's consider gravity-assist maneuvers using the planet Jupiter as an example. Given that Jupiter is the largest planet in the solar system, it has the strongest gravity field, and so the effect it has on the dynamics of a passing spacecraft is very significant. Let's imagine our spacecraft wandering through interplanetary space, traveling out from Earth toward Jupiter. We also suppose that, in this interplanetary cruise, the spacecraft is in an elliptical orbit around the Sun, so that it is the Sun's gravity that governs its motion. However, when the spacecraft is about 40 million kilometers (25 million miles) from Jupiter, the gravity field of Jupiter begins to be about the same as that of the Sun. Beyond this point, as the spacecraft closes in on Jupiter, the force of Jupiter's gravity increases, and once it is within, say, 10 million kilometers (6 million miles) of Jupiter, the Sun's influence is negligible. At this position, point A in Figure 4.2, the spacecraft is effectively falling toward Jupiter, moving in a (more or less) straight line with a speed given by V_{in} relative to Jupiter. In this part of the flight, while governed by Jupiter's gravity field, the spacecraft performs the classic hyperbolic trajectory as described above. It reaches a maximum speed at closest approach (point B), and then climbs away arriving at point C traveling effectively in a straight line once again with a speed V_{out} relative to Jupiter. From our discussion above about the spacecraft's speed on the hyperbola, we know that V_{in} and V_{out} are the same, so we don't seem to have gained anything from the maneuver! Remember, however, that these are *speeds relative to Jupiter.* The thing we have forgotten is that Jupiter itself is moving along its orbit around the Sun at about 13 km/sec (8 miles/sec) relative to the Sun, and this makes all the difference.

In Figure 4.2, the speeds of objects are represented by arrows. The orientation of the arrow represents the direction in which the object is moving, and the length of the arrow indicates how fast it is going: long arrows denote fast objects, and short arrows denote slow objects. These arrows are called *velocity vectors,* which are useful tools in the analysis of this type of problem. Incidentally, in Chapter 3 we used force vectors, although we did not describe them as such, in the discussion of orbit perturbations, where the direction of an arrow indicated the direction of a force, and the length of the arrow indicated how much force was applied. There are lots of

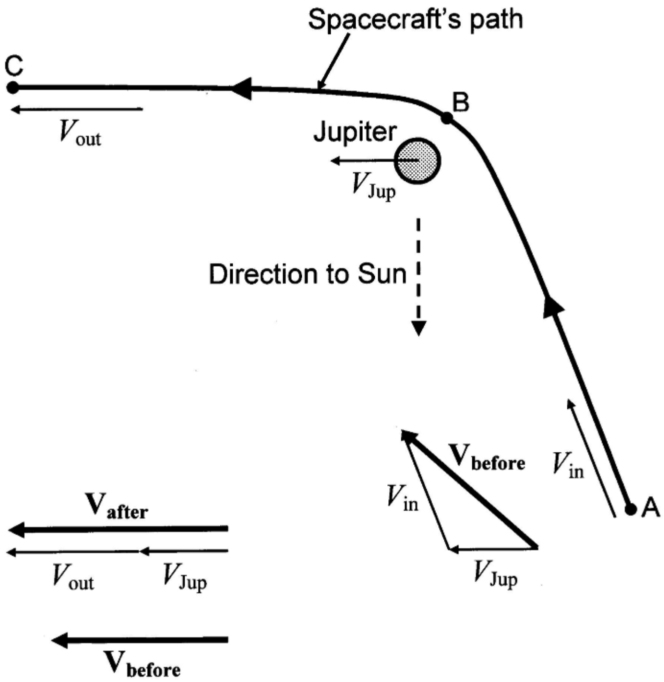

Figure 4.2: A schematic of a Jupiter gravity-assist maneuver.

objects in physics and dynamics that require two pieces of information—direction and magnitude—to describe them fully, and these are all described as vectors by scientists and engineers. For example, there are position vectors, velocity vectors, force vectors, torque vectors, and others, but I'm getting away from the point here.

Getting back to our gravity-assist maneuver, what defines the orbit around the Sun before the swing-by is the position and velocity of the spacecraft relative to the Sun before it enters Jupiter's gravitational influence. Let's suppose that point A in Figure 4.2 is now right at the edge of Jupiter's sphere of influence, approximately 40 million kilometers (25 million miles) out from the planet. If V_{in} again represents the incoming velocity of the spacecraft relative to Jupiter at A, and V_{Jup} is the velocity of Jupiter on its orbit around the Sun, then we can calculate the velocity of the spacecraft relative to the Sun before the encounter. This is denoted by $\mathbf{V}_{before}$ in the figure, and it is the sum of the velocity of the spacecraft relative to Jupiter and the velocity of Jupiter relative to the Sun. Given that the arrows, or vectors, representing these velocities are not parallel, we have to use a *velocity vector diagram* to do this sum, and this is shown as the triangle in the lower right of the figure. The important thing to note is that $\mathbf{V}_{before}$

represents the velocity of the spacecraft on its elliptical orbit around the Sun before meeting Jupiter.

Working out the corresponding velocity of the spacecraft relative to the Sun afterward is a little easier, as in this case the velocities V_{out} and V_{Jup} are parallel to each other, so $\mathbf{V}_{\mathbf{after}}$ is just a simple arithmetical sum. This is shown in the lower left in Figure 4.2, opposite the vector triangle. V_{out} and V_{Jup} need not necessarily be parallel to each other, but I have devised the gravity-assist geometry so that they are, to make things a little easier to visualize. Remember that $\mathbf{V}_{\mathbf{after}}$ represents the spacecraft's velocity relative to the Sun after the gravity assist, and so defines the subsequent orbit around the Sun after the encounter with Jupiter. In the bottom left of Figure 4.2, the magnitudes of the Sun-relative speeds before and after the encounter—$\mathbf{V}_{\mathbf{before}}$ and $\mathbf{V}_{\mathbf{after}}$—are compared, and it is easy to see that the spacecraft has gained speed.

The next question to ask is, Where has that speed gain come from? The answer is that the spacecraft's gain is Jupiter's loss. The planet has tugged on the spacecraft, to give it a significant boost in speed, while at the same time the spacecraft has exerted an opposite tug on Jupiter, causing it to lose speed. However, given the mass of the spacecraft compared to the huge mass of Jupiter, the effect on Jupiter is immeasurably small, although it is measurable if you wait long enough; in the words of a National Aeronautics and Space Administration (NASA) press release about the Voyager Jupiter swing-by, "The position of Jupiter will change by about 1 foot every trillion years"!

Figure 4.3 shows a remarkable plot of the Sun-relative speed of Voyager 2 during its interplanetary cruise to the outer planets. The "blips" in the curve at around 5, 10, 20 and 30 astronomical units (AU) correspond to the gravity-assist maneuvers at Jupiter, Saturn, Uranus, and Neptune, respectively. You may recall from Chapter 1 that 1 AU is equal to the mean Earth–Sun distance. These maneuvers maintained the speed of the spacecraft well above that needed to escape the Sun, which is indicated by the broken line.

We can get a feel for the way gravity-assist maneuvers work by proposing a useful analogy, in the form of a rather unusual experiment involving a double-decker bus. Firstly, I would definitely recommend that you do not attempt to perform the experiment, as I would not wish to be responsible for any resulting injuries—and secondly, it allows the rather unusual notion of having a picture (see Figure 4.4a) of a double-decker bus in a book about spaceflight! Those of you who are familiar with the old-style double-decker bus know that the entrance is via a step-up, open platform at the back of the bus as shown in Figure 4.4, with a vertical handrail for passengers to hang onto to prevent them from falling out while the bus is moving. The vertical handrail plays the role of the planet Jupiter in the experiment!

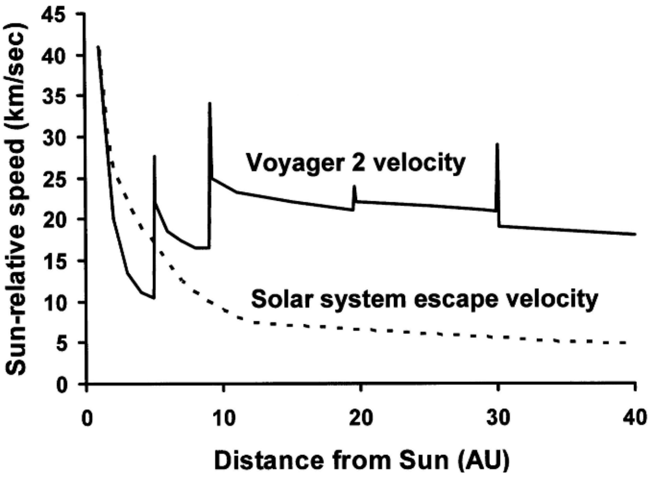

Figure 4.3: The speed of the Voyager 2 spacecraft relative to the Sun during its 12-year mission to the outer planets. (Figure compiled from data courtesy of Steve Matousek, National Aeronautics and Space Administration [NASA]/Jet Propulsion Laboratory [JPL]—Caltech.)

The layout of the experiment is illustrated in Figure 4.4b, which shows a view of the bus looking down from above. The vertical handrail is positioned at the rear corner of the bus, where the open platform entrance is located. Note that the entrance has been switched to the right-hand side of the bus, to allow us to compare the geometry more easily with Figure 4.2. A person, shown rather unimaginatively as a blob with an extended arm and hand, is shown standing on the road, swinging on the handrail. Surprisingly, this rather unlikely arrangement gives a good depiction of what happens in a gravity-assist maneuver, as discussed above. Each of the items shown in Figure 4.4b represents some part of the real thing. For example, as mentioned above, the vertical handrail represents the planet Jupiter, where its movement over the ground, as the bus moves forward, represents the movement of the planet in its orbit around the Sun. The person swinging on the rail represents the spacecraft, and the force in the person's arm as he swings on the rail plays the role of the gravitational attraction between the planet and the spacecraft. Now we can play the experiment two ways—first with the bus stationary and then with it moving.

The first task for our willing helper is fairly simple then, as we would just like them to run up to the rear of the stationary bus at, say, 10 mph (4.5 m/sec)—see plan view in Figure 4.5—grab hold of the vertical hand rail at point 1, and then release his grip at point 2. This simply has the effect of changing the direction of his run from the vector V_{in} to the vector V_{out} as

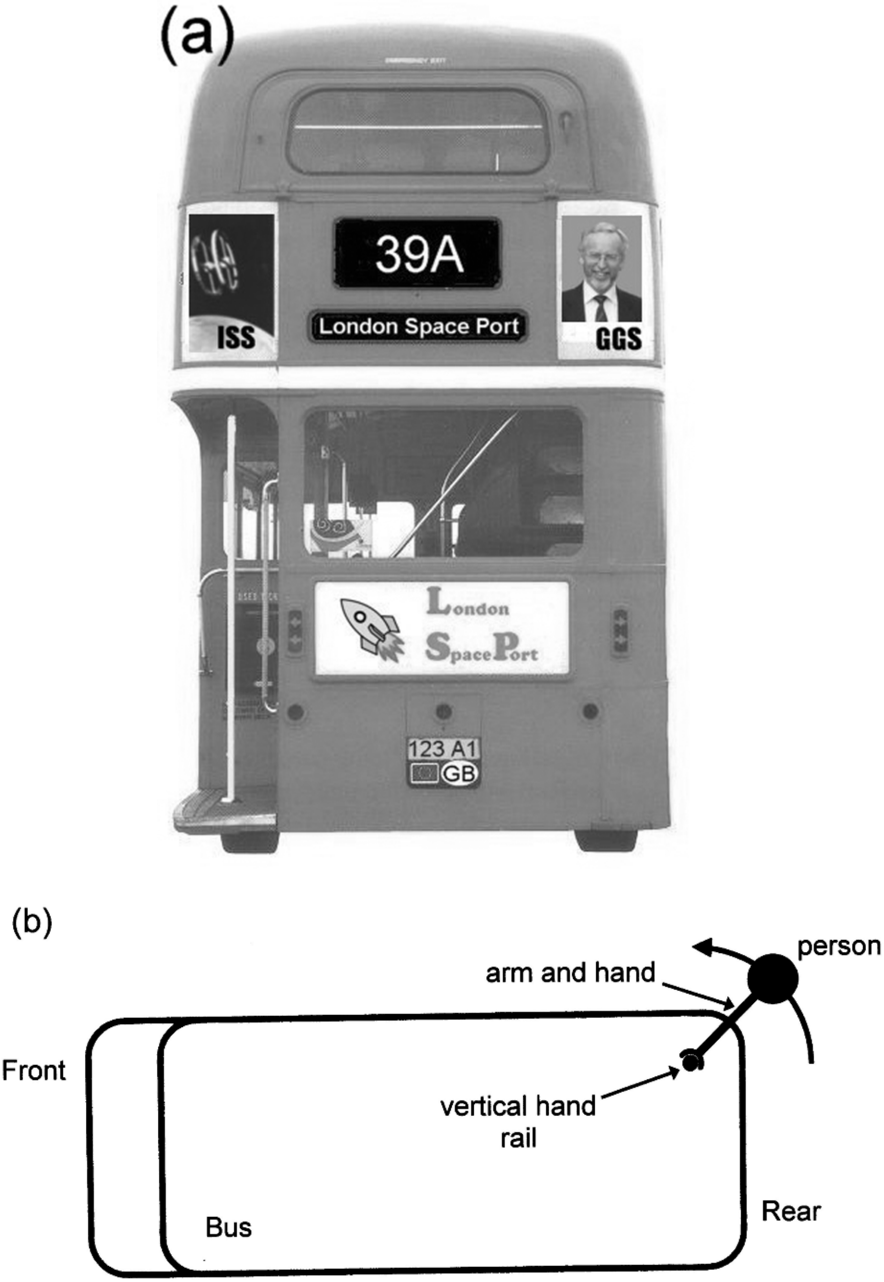

Figure 4.4: (a) The back of a bus! Note the vertical handrail in the open, step-up platform entrance. (b) Plan view of bus, showing relevant features of the thought experiment described in text.

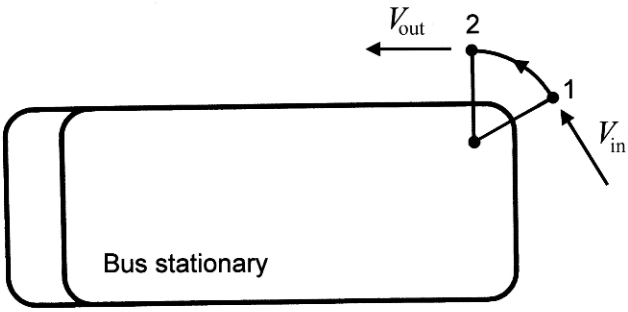

Figure 4.5: Depiction of a hyperbolic swing-by, using the bus analogy.

shown, without effectively changing his speed. This parallels a hyperbolic swing-by trajectory, where the incoming and outgoing speeds relative to the planet are the same but the direction of travel of the spacecraft is changed.

To go any further with the experiment, we have to suppose that our helper is fairly athletic, and has good hand–eye coordination, as we will now expect them to grab the handrail while the bus is moving. Not wishing to make this too difficult, we'll constrain the speed of the bus to 10 mph. This situation is now shown in plan view in Figure 4.6. Again, the runner grabs the rail at point 1 and releases it at point 2. To make a proper parallel with the

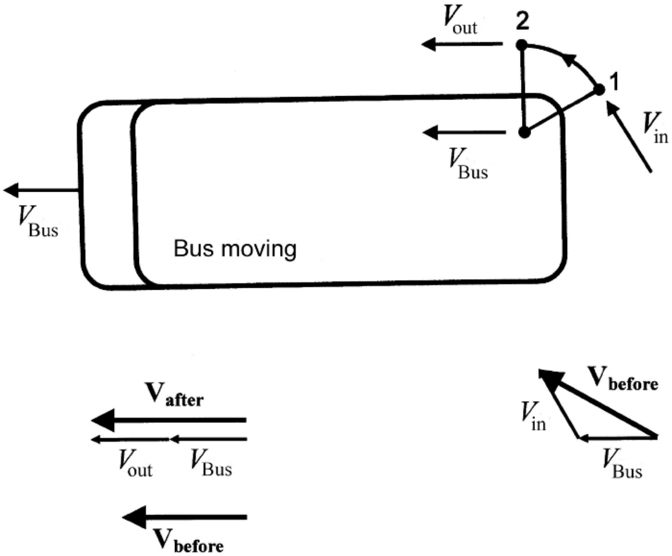

Figure 4.6: Depiction of a gravity-assist maneuver. The speed of the runner over the ground is increased by the movement of the bus. Compare with Figure 4.2.

spacecraft gravity assist, the runner has to perform his swinging movement on the rail in such a way that it looks the same as the stationary bus swing from the perspective of someone standing on the moving bus platform. But then, perhaps this is one technical detail too far? The important thing to take away from the moving bus analogy is an intuitive feel that once the runner grabs the rail, he acquires additional speed over the ground from the movement of the bus. If you imagine grabbing the moving handrail yourself, you can almost feel the stress in your arm, tending to pull the socket, as your body mass is accelerated by the bus! The force in your arm will also act on the bus in the opposite sense, tending to slow it down a little, in the same way as we saw Voyager taking a tiny amount of orbital speed from Jupiter. The speeds over the ground of the runner $\mathbf{V_{before}}$ and $\mathbf{V_{after}}$ are worked out at the bottom of Figure 4.6, in an analogous way to that done in the swing-by case in Figure 4.2, confirming an increase in speed. Note that the speed over the ground in this analogy corresponds to the speed of a spacecraft relative to the Sun in the gravity-assist maneuver.

This analogy presents a situation in which you can at least imagine experiencing a boost in speed in a way that is similar to what happens in a gravity assist.

Orbits Around Small, Irregularly Shaped Bodies

Recently there has been a lot of interest in spacecraft missions that visit the smaller objects in the solar system, such as asteroids and comets. These types of objects are essentially debris scattered across interplanetary space, which are fragments left over from the processes that formed the Sun and planets. And therein lies their attraction as targets of scientific interest for spacecraft probes.

Asteroids, sometimes called minor planets, are usually solid bodies which vary in size from around 900 km (560 miles) in diameter to tiny boulders a few meters across. The majority of these objects travel in orbits between Mars and Jupiter, although many of the smaller objects can be found almost anywhere in the inner solar system. Comets are also small objects, typically about 1 to 10 km in diameter, composed mostly of ice and dust. The current view is that these "dirty snowballs" originate from a region of space distant from the Sun, called the *Oort cloud*, around tens of thousands of AU away from the Sun. Periodically, a comet will be knocked out of the cloud by the gravitational disturbances of a passing star, causing it to fall into the inner solar system. As it does so, once it is closer than about 5 AU from the Sun, the ice begins to react to solar heating, causing plumes of gas and dust. This produces the characteristic appearance of comets—a compact nucleus and a long tail, shining magnificently by reflected sunlight. In past times, they were often greeted with a mixture of awe and suspicion, and were sometimes regarded as an omen of some impending disaster. Fortunately, the public is now better informed, although a strong reaction can still be evoked by a bright comet.

These small bodies are believed to contain crucial clues to aid understanding of the origin and early evolution of our solar system, which is why there has been growing interest in sending spacecraft to investigate them. The first orbital mission around an asteroid was achieved by the Near Earth Asteroid Rendezvous (NEAR) Shoemaker spacecraft, which entered orbit around its target asteroid Eros in February 2000. The spacecraft was not designed to land on the asteroid's surface, but the mission was finished off in February 2001 with a rather ad hoc descent and touchdown on Eros's surface, making for a first in astronautical history. The other high-profile mission in progress at the time of this writing is the European Space Agency's Rosetta mission to orbit and land on a comet. Rosetta's rendezvous with its intended target will not occur until May 2014. On arrival, the Rosetta spacecraft will enter orbit around the small comet 67P/Churyumov-Gerasimenko, finally deploying a small instrument package to make a soft landing on the comet's nucleus.

If we think about orbits around asteroids and comets, they do not fall into the ideal-orbit category discussed in Chapters 2 and 3. You may recall that a so-called ideal orbit is one that results from the motion of a spacecraft in a pure inverse square law gravity field. This produces the familiar trajectory shapes called conic sections: the circle, the ellipse, the parabola, and the hyperbola. The main reason why orbits around asteroids and comets do not fall into this category is that these objects are usually irregular in shape, so that their gravitational fields are nothing like that described by Newton's inverse square law. This affects the orbital motion significantly.

You may also remember from Chapter 2 that the motion of a spacecraft in an ideal circular or elliptical orbit takes place in a fixed plane, and is periodic in the sense that after each orbital revolution the vehicle returns to the same position with the same orbital speed. For the motion of a spacecraft around an irregularly shaped body, neither of these things is true. Figure 4.7 shows an orbit around an asteroid where the track of the spacecraft around the object does not join up, and in fact varies on each orbital rotation. This departure from ideal motion is particularly significant for close orbits around the object. Given that the orbit track is not repeated on each revolution around the object, it is possible that, after a number of orbits, the spacecraft may impact the surface. Care is required to choose a close orbit

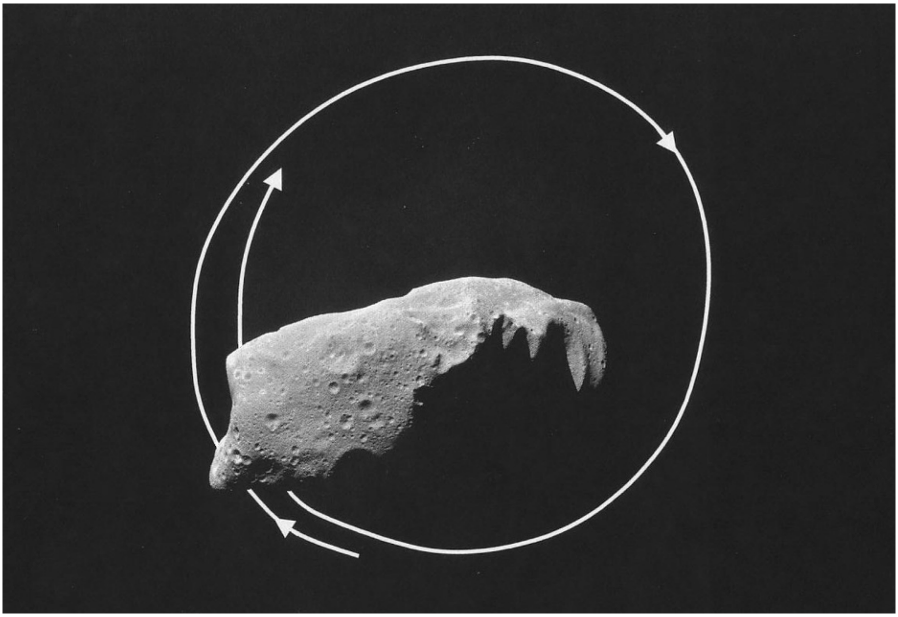

Figure 4.7: A close orbit around a small, irregularly shaped body is perturbed by the nonspherical distribution of mass in the asteroid. The picture is of the asteroid Ida, imaged by the Galileo spacecraft in August 1993. (Backdrop image courtesy of NASA/JPL-Caltech.)

that has long-term stability. On the other hand, if the spacecraft were to orbit some distance away, say 20 asteroid diameters, then the irregularities in the bodies shape become less influential, and the spacecraft's motion approximates well to an ideal orbit.

For missions like Rosetta, the process of deploying a lander on a comet can be quite torturous, given that the spacecraft has to be in fairly close orbit around the nucleus to successfully achieve this deployment. The first issue for the spacecraft designer is that the surrounding environment and the size of the comet are not known before the spacecraft gets there. This means that the spacecraft systems and payloads have to be designed to successfully and safely accommodate a range of conditions. The fact that the size, mass, and shape of the comet are unknown at the spacecraft design stage means that a gradual approach to a close orbit is required. As a consequence the spacecraft is inserted into an initial orbit with a large radius compared to the size of the comet.

For the sake of argument, let's suppose the comet nucleus is a fairly average 5 km (3 miles) in diameter, with a density about that of water. If we make the initial orbit radius equal to about 20 comet diameters, then the orbit would approximate well to an ideal circular orbit with a radius of around 100 km (62 miles). The speed of the spacecraft in this initial orbit is about 20 cm/sec (8 inches/sec), which compared to a brisk walking speed of around 4 mph or 1.8 m/sec (5.9 feet/sec) is an amazingly sedate speed for a spacecraft! This gives a first insight into the way mission orbit activities are carried out in proximity to a comet; it can be describe as a low-energy environment where things move slowly, and take a long time to happen. The other implication of this low orbital speed is the precision with which the entry into orbit has to be made. In this circular orbit the escape velocity is only 30 cm/sec (1 ft/sec), compared to about 11 km/sec for Earth, so an error in speed in excess of just 10 cm/sec in orbit insertion speed would result in the spacecraft's disappearing from the comet altogether!

The next job for the mission analysis team is to try to lower the orbit radius to, say, 2 comet diameters (10 km) to allow the lander to be detached from the spacecraft and begin its descent to the surface. However, to fly the spacecraft that close to an irregularly shaped comet would require the team to have detailed knowledge of the gravity field to facilitate accurate prediction of the spacecraft's trajectory. As mentioned above, without this knowledge it is possible over a number of orbital revolutions for the spacecraft to impact the surface. To acquire this knowledge, the comet is examined from the initial high orbit using imaging sensors to acquire detailed information about its shape. At the same time, the spacecraft is tracked precisely in order to detect the small perturbing forces that provide a clue to how the object's gravity field differs from an inverse square law. These data, on shape and perturbations, allow a first estimate of the gravity field of the comet to be made by the mission analysis team. With this information, the orbit radius can be lowered further to, say, 5 comet diameters (25 km), where the process can be repeated, allowing a further refinement in knowledge of the gravity field. Finally, once the team is confident that it has sufficient knowledge of the gravity field, the spacecraft can be inserted into its final close orbit, from which the lander can be deployed. The orbital speed in the 10 km radius orbit is around 60 cm/sec (2 feet/sec), still much slower than our brisk walking pace. In a real project situation, the choice of this orbit radius is difficult to pin down. On the one hand, it has to be small enough so that the descent time of the lander is not too long, but it also has to be large enough so that the orbiting spacecraft is not damaged or contaminated by the near-comet environment. The near-comet space is a dynamic environment, particularly when the comet is close to the Sun. Solar heating causes the surface of the comet to evaporate in plumes of gas and dust, from which the orbiting spacecraft needs to keep a distance. There is always a conflict between the engineers and scientists in these situations. The engineers want to keep the spacecraft a safe distance from hazards, whereas the scientists want it to be "where the action is." Clearly a compromise has to be struck between the value of the science and the risk to the spacecraft.

Let's suppose the lander is an instrument package with landing legs, having a total mass of, say, 100 kg. The next task is to detach it from the orbiting spacecraft, so that it can begin its descent to the surface. The simplest way of doing this is to push the lander out the back of the spacecraft, with an ejection speed equal to the spacecraft's forward orbital speed (how this is done is not really relevant to the story here, but it would probably be most easily done with a calibrated spring mechanism). If the backward speed of the lander matches the forward speed of the spacecraft—around 60 cm/sec in this case—then the lander is left momentarily motionless above the comet's surface. The comet's weak gravity will then cause it to fall steadily toward the surface. Because the gravity of the comet is so weak, this trip to the surface takes quite some time, around 4.4 hours in our example, and when the lander finally approaches the surface its descent speed is only around 1.6 m/sec (just a shade less than our brisk walking speed of 4 mph). On final approach to the surface, contamination of the surface by spraying it with rocket engine exhaust gases may not be a good idea from the point of view of the science objectives of the mission. This may prevent the use of a rocket engine to slow the descent, so that the structure of the lander may need to be able to withstand the 1.6 m/sec impact speed.

Perhaps the major issue with the touchdown is preventing the lander from bouncing back off the surface, and possibly going back into orbit. This is a real possibility, as the 100-kg-mass spacecraft will weigh only about $1/10^{th}$ of a Newton on the surface, which is about $1/10^{th}$ the weight of a small apple! And of course the nature of the surface, whether it's bouncy, or sticky, or somewhere in between, is unknown until touchdown. To prevent such a bounce happening, the lander will either have to fire an upward directed rocket motor or mechanically grapple the surface somehow, on touchdown.

Figure 4.8 shows the European Space Agency's Rosetta lander after what is hoped to be a successful touchdown on comet 67P/Churyumov-Gerasimenko in around 2014. The complexities of the orbital aspects of such a mission are not to be underestimated, where the central body is no longer a nice uniform sphere, but has a shape resembling a potato.

Figure 4.8: Artist's impression of the Rosetta lander after a successful touchdown on comet 67P/Churyumov-Gerasimenko. (Image courtesy of the European Space Agency [ESA].)

Halo Orbits Around Lagrangian Points

In earlier chapters, we described a variety of orbits around Earth, but in this section we discover that it is possible for a spacecraft to orbit around a point in space where there is no mass! This rather fascinating state of affairs requires some explanation.

The Three-Body Problem

The story starts in the 1770s with an Italian-French mathematician named Joseph Lagrange. Note that this is about 100 years after Newton gave the world his revelations about how objects move in gravity fields, giving the world's scientists and mathematicians potentially centuries of research work

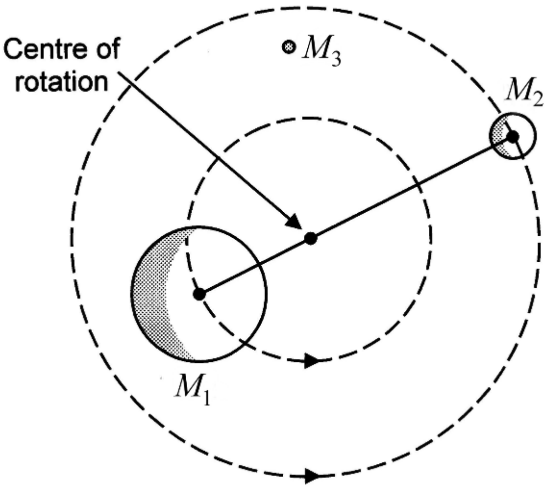

Figure 4.9: The circular restricted three-body problem. The two massive bodies M_1 and M_2 move in circular orbits around each other. The third body M_3 is of negligible mass and moves under the gravitational influence of M_1 and M_2.

to do to reveal their full significance. Lagrange was using Newton's laws to study something called the *three-body problem*, which, as the name suggests, is the investigation of how three massive bodies move around each other under gravity. Unfortunately, he found the problem to be complex and unsolvable, which it remains to this day. However, his work was not entirely fruitless. In his attempts to make the problem more amenable to solution, Lagrange examined simplified versions of the full three-body problem and in the process discovered *Lagrangian points*, which, as we will see, are relevant to modern spacecraft mission design.

The simplified version that Lagrange looked at is shown in Figure 4.9, and involves two massive bodies, M_1 and M_2, in circular orbits around each other, and a third much smaller body M_3 moving along a trajectory influenced by the gravity fields of its two larger neighbors. This setup is referred to as the *circular restricted three-body problem* (CRTBP). The important thing to note here is that the third body is so small in mass that it has negligible effect on the motion of the two larger bodies. So how do two massive bodies rotate around each other in circular orbits? If they are of equal mass, then they rotate about a point that is halfway between their centers (Fig. 4.10a). If their masses are dissimilar, they rotate about a point that is closer to the larger object (Fig. 4.10b). This point about which the rotation takes place is referred to as the *barycenter* of the system. For example, in the Earth–Moon system, the mass of Earth is about 81 times

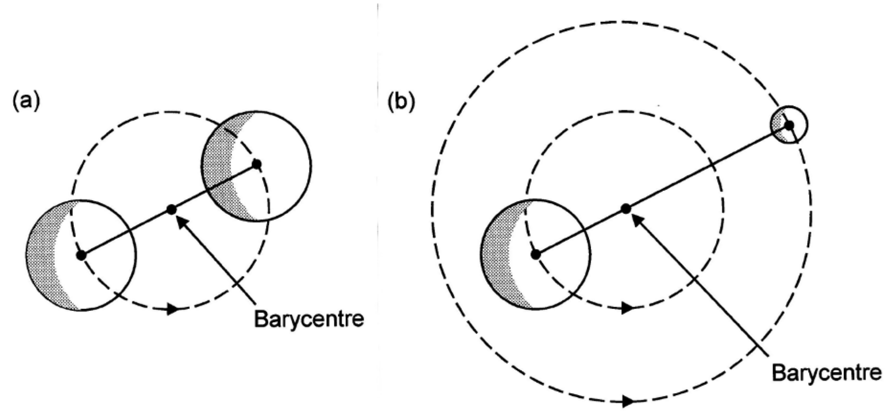

Figure 4.10: Massive bodies moving in circular orbits around each other rotate about their barycenter.

larger than that of the Moon, so the barycenter about which Earth and the Moon orbit is only about 5000 km (3100 miles) from Earth's center—actually beneath Earth's surface.

If we return to Lagrange's simplified problem, illustrated in Figure 4.9, there are a number of good examples of this type of system that are relevant to modern spacecraft mission design. The most obvious of these, from the 1960s, is an Apollo spacecraft on its way to the moon. In this example, Earth and the Moon represent the larger bodies in (nearly) circular orbits around each other, and the spacecraft represents the third body of negligible mass. Another example of a CRTBP with wide applications is the Sun–Earth–spacecraft system.

In his mathematical exploration of the CRTBP, Lagrange discovered five points in the rotating system where the third body (of negligible mass) could remain stationary relative to the two larger bodies. These *equilibrium points* are illustrated in Figure 4.11, and are referred to as Lagrangian points L_1, L_2, L_3, L_4 and L_5 in Lagrange's honor. When looking at Figure 4.11, it is important to realize that the relative geometry between the large bodies and the Lagrangian points remains fixed, and rotates about the system's barycenter.

Orbits About Lagrangian Points

What is the relevance of all this to spacecraft mission design? To focus the discussion, let's look at the situation where the two larger bodies are the Sun and Earth, and the smaller body is a spacecraft. Each Lagrangian point is a place where the forces of gravity and those due to the rotation of the system add up to nothing (we'll return in a moment to discuss what we mean by

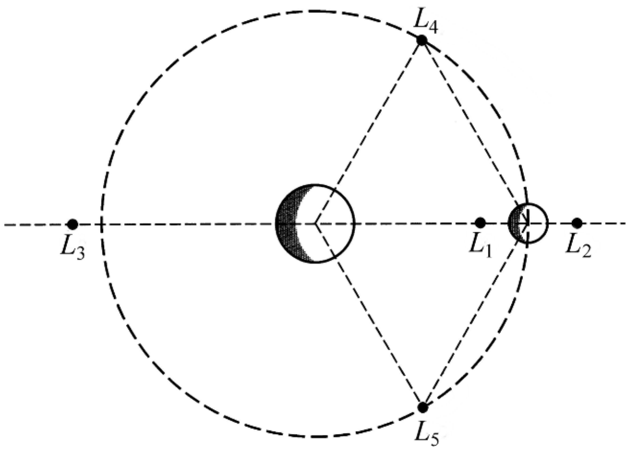

Figure 4.11: The locations of the Lagrangian points L_1, L_2, L_3, L_4 and L_5 relative to the two larger masses in a rotating system.

"forces due to rotation"), which gives it the characteristic of being an equilibrium point. If you locate a spacecraft at any one of these points, it will remain there. To further focus the discussion, let's consider the L_1 and L_2 points in the Sun–Earth system, as these have attracted most interest in terms of spacecraft applications. In this system, the L_1 point is located approximately 1,500,000 km (930,000 miles) away from Earth, in the direction of the Sun, and the L_2 point is a similar distance from Earth in the opposite direction (Fig. 4.11). Positioning a spacecraft at the L_1 and L_2 points seems like a straightforward affair, apart from one detail. A more detailed look at the mathematics tells us that these are points of *unstable equilibrium,* which means that if the spacecraft is disturbed by the slightest perturbation, it will move away from the Lagrangian points. This state of unstable equilibrium is a bit like trying to balance a small metal ball, like a ball bearing, on top of a smooth dome-shaped surface. With enough care, you may be able to balance the ball at the summit of the dome, but the slightest disturbance will cause it to roll away down the slope. And so it is with a spacecraft. However, with the help of the spacecraft's propulsion system, it is possible to regain stability, and furthermore to control the spacecraft in an orbit around the Lagrange point. How the spacecraft orbits a massless point is most easily explained by considering the motion about the L_1 point in the Sun–Earth–spacecraft system.

Strictly speaking, the L_1 point is a location in the rotating system where the gravitational and rotational forces sum to zero. I think we are fairly happy thinking about the forces of gravity of the Sun and Earth acting upon the spacecraft, but what do we mean when we talk about rotational forces? The Sun–Earth system rotates only slowly, about 1 degree per day due to the Earth orbiting the Sun with a period of 1 year, so whatever forces there are due to rotation, they must be small. But in this instance, they do nevertheless play an important role.

Perhaps the best way to think about rotational forces is to imagine yourself on a small merry-go-round, the kind you see in a child's playground. Maybe as a child you stood on one of these, holding on to the safety rails, while a friend spun it up to perhaps an uncomfortably high speed. In this situation, you certainly get a good impression of a rotational force, as this is the force you feel tending to throw you off the merry-go-round. This outward directed force you experience in a rotating system—the merry-go-round in this case—is referred to as *centrifugal force*. To prevent yourself from being hurled off the merry-go-round, you have to hold on tightly to the safety railings. You are able to remain standing on the same spot on the merry-go-round because the force in your arms pulling toward the center of the merry-go-round balances the centrifugal force tending to throw you off.

As it happens, this is a remarkably good analogy to describe the manner in which the spacecraft can remain "standing on the same spot" at the L_1 point in the rotating Sun–Earth system. The force tending to pull the spacecraft toward the Sun is the Sun's gravity, and this is analogous to the force in your arms as you hold on tightly to the merry-go-round's safety rails. The force tending to pull the spacecraft outward is predominantly centrifugal, generated by the rotation of the system, but there is also a small contribution from Earth's gravity. There is a balance of forces on the spacecraft—solar gravity inward, and centrifugal force plus Earth gravity outward—allowing the spacecraft to remain stationary at the L_1 point. To be precise, 97% of the outward force balancing solar gravity is centrifugal, and only 3% is Earth's gravity.

But we still have not addressed how a spacecraft can orbit the L_1 point. In Figure 4.12 we see that if the spacecraft is displaced along the line joining the Sun and Earth to point 1 or point 2, then the sum of gravity and rotational forces is no longer zero, and the vehicle will tend to move away from the Lagrangian point. This is an expression of the instability of the L_1 point that we referred to earlier.

However, there is a *surface of stability* upon which the spacecraft can orbit the L_1 point. This is the surface at right angles to the Sun–Earth line, which passes through the L_1 point, as shown in Figure 4.12. It is slightly curved, as shown in a rather exaggerated way in the figure, but it can be thought of as a plane surface in which the orbital motion takes place. Now we can see that if

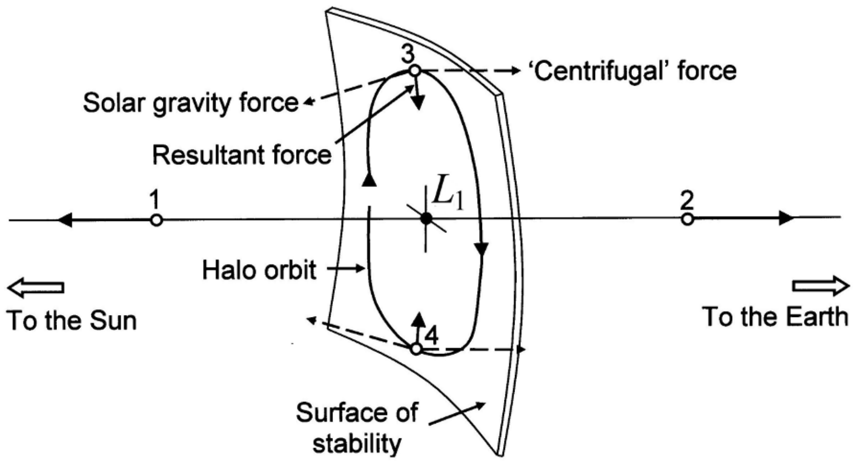

Figure 4.12: An illustration of how a spacecraft can orbit a massless Langrangian point.

the spacecraft is located at point 3 or point 4, the sum of the Sun's gravity force in one direction, and centrifugal force (with a little bit of Earth gravity) in the other, produces a resultant force directed toward the L_1 point. In fact, it is easy to see that this L_1 directed force occurs at any point on the illustrated halo orbit in the figure, thus allowing an orbital motion around the massless L_1 point. The shape of the orbit around L_1 is certainly not a conic section, in general, and can in fact be a weird variety of looping curves referred to as a *Lissajous orbit.* Also, as we have mentioned already, the spacecraft has to use its propulsion system to tweak the motion to ensure long-term stability of the orbit.

A similar explanation of the orbital motion around the L_2 point can be argued, with a balance between an inward directed force composed of Sun and Earth gravity and a outward directed centrifugal force. The latter force is slightly larger at the L_2 point since it is further away from the center of rotation.

As for the idea of a *resultant force,* the configuration of force vectors at point 3 in the figure is similar to the forces acting on an arrow when it is fired from a bow. In Figure 4.13 the force vectors actually acting on the arrow are the tension forces in the string on either side of the arrow. But the sum of these—the resultant force—is actually directed along the arrow, and produces the acceleration that makes it fly.

Why are we interested in this concept? The idea of using Lagrangian point orbits for spacecraft is not a new one. A halo orbit around the L_1 point, between Earth and the Sun, is an ideal location for a spacecraft with a Sun-

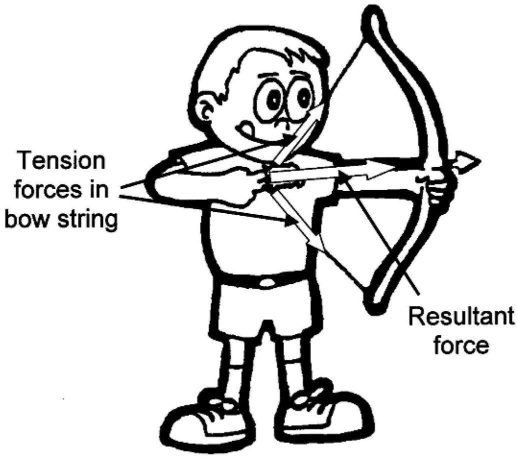

Figure 4.13: The tension forces in a bow string produces a resultant force along the arrow.

viewing payload, as such a spacecraft has an uninterrupted view of the Sun. Examples of solar observatory spacecraft that have resided at L_1 are the SOlar and Heliospheric Observatory (SOHO) and the Advanced Composition Explorer (ACE). Conversely, the L_2 point is further away from the Sun than the Earth, above Earth's night side, and this point is a good location for space telescopes. The sky is not obscured by Earth as it is for the Hubble Space Telescope in its low Earth orbit; Earth subtends an angle of only about half a degree from the L_2 point. The Wilkinson Microwave Anisotropy Probe (WMAP) spacecraft is an example of a space observatory that has used a L_2 point orbit. Looking to the next generation of space telescope beyond Hubble, the James Webb Space Telescope (JWST) is destined for a L_2 halo orbit around the year 2013.

Bored with Orbits?

We seemed to have spent quite a time talking about orbital motion, one way or another, over the last four chapters. In Chapter 5 let's take a break and have a look at how we use rocket-powered launchers to get our spacecraft off the ground and into orbit.