Heights, the Geopotential, and Vertical Datums

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1. Introduction

With the Global Positioning System (GPS) now providing heights almost effortlessly, and with many national and international agencies in different regions of the world re-considering the determination of height and their vertical networks and datums, it is useful to review the fundamental theory of heights from the traditional geodetic point of view, as well as from the modern standpoint which addresses the centimeter to sub-centimeter accuracy that is now foreseen with satellite positioning systems.

The discussion assumes that the reader is somewhat familiar with physical geodesy, in particular with the foundations of potential theory, but the development proceeds from first principles in review fashion. Moreover, concepts and geodetic quantities are introduced as they are needed, which should give the reader a sense that nothing is a priori given, unless so stated explicitly.

2. Heights

Points on or near the Earth's surface commonly are associated with three coordinates, a latitude, a longitude, and a height. The latitude and longitude refer to an oblate ellipsoid of revolution and are designated more precisely as *geodetic latitude and longitude*. This ellipsoid is a geometric, mathematical figure that is chosen in some way to fit the mean sea level either globally or, historically, over some region of the Earth's surface, neither of which concerns us at the moment. We assume that its center is at the Earth's center of mass and its minor axis is aligned with the Earth's reference pole. The height of a point, P, could refer to this ellipsoid, as does the latitude and longitude; and, as such, it designates the distance from the ellipsoid to the point, P, along the perpendicular to the ellipsoid (see Figure 1); we call this the *ellipsoidal height*, h_P.

However, in most surveying applications, the height of a point should refer to mean sea level in some colloquial sense, or more precisely to a *vertical datum* (i.e., a well-defined reference surface for heights that is accessible at least at one point, called the *origin point*). We note that the ellipsoid surface is not the same as mean sea level, deviating from the latter on the order of 30 m, with maximum values up to 110 m, if geocentrically located. On the other hand, the ellipsoid may be identified as the "vertical datum" for ellipsoidal heights. In this case accessibility is achieved indirectly through the assumption that the ellipsoid is defined in a coordinate frame established by the satellite observations that yield the three-dimensional coordinates of a point.

A comprehensive discussion of heights with respect to the traditional vertical datum, the geoid, cannot proceed without introducing the concept of *potential*; this is the topic of the next section.

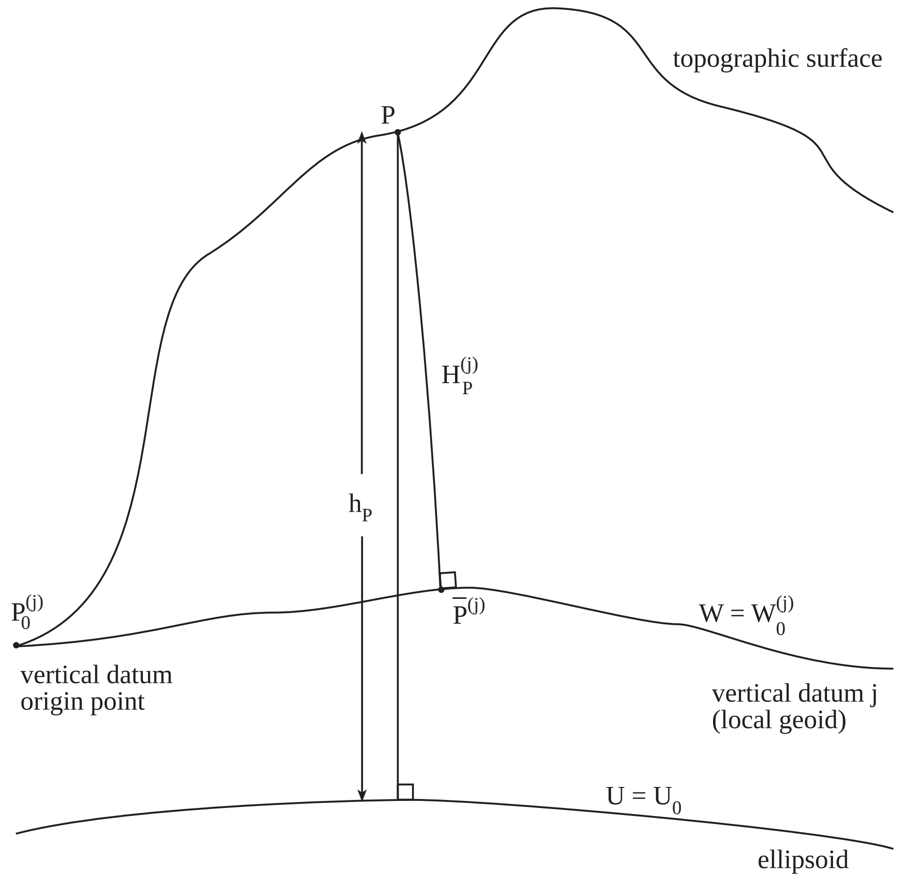

Figure 1: Ellipsoidal height versus orthometric height with respect to vertical datum, j.

2.1 Geopotential and Geoid

The *geopotential* is the gravitational potential generated by the masses of the Earth, including its atmosphere. In addition, the potential from other masses in the solar system may be considered separately, especially that of the sun and moon, as they introduce a time-varying field due to their apparent motions relative to the Earth. The geopotential, V_e, is expressed most conveniently in terms of spherical harmonic functions (an alternative formulation in terms of ellipsoidal harmonics is also sometimes used):

$$V_e(r,\theta,\lambda) = \frac{kM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{R}{r}\right)^{n+1} C_{nm} \, \bar{Y}_{nm}(\theta,\lambda) \,, \tag{1}$$

where (r,θ,λ) are the usual spherical polar coordinates (radial distance, co-latitude, longitude–these are not the geodetic coordinates mentioned above), kM is the product of Newton's gravitational constant and Earth's mass (including the atmosphere), R is a mean Earth radius, C_{nm} is a constant coefficient for degree n and order m, $\bar{Y}_{nm}$ is a fully-normalized, surface spherical harmonic function:

$$\bar{Y}_{nm}(\theta,\lambda) = \begin{cases} \bar{P}_{nm}(\cos\theta)\cos|m|\lambda\,, & m \ge 0\,; \\ \\ \bar{P}_{n|m|}(\cos\theta)\sin|m|\lambda\,, & m < 0\,; \end{cases} \tag{2}$$

and $\bar{P}_{nm}$ is a fully-normalized Legendre function of the first kind. The series expression (1) is the solution to a boundary-value problem for the potential and holds only if the point with coordinates (r,θ,λ) is in free space (no mass). This is just a *necessary* condition; whereas, the convergence of the series to the true gravitational potential is guaranteed only for points outside a sphere enclosing all masses (which represents a *sufficient* condition). In practice we have available a truncated version of the series ($n \le n_{max}$) that is used to approximate the potential at any point above the Earth's surface; this space exterior to the Earth, moreover, is approximated as being "free space". The effect of the atmosphere on the potential is, in fact, substantial, approximately 54 m^2/s^2 near the surface, but it is fully compensated to first order (Sjöberg, 1999) by mathematically "moving" the atmosphere radially inside the Earth.

Because of Earth's rotation, static gravimeters on the Earth experience a centrifugal acceleration that (because of Einstein's Equivalence Principle) cannot be distinguished from the effect of gravitation. In geodesy, we term the combination of centrifugal and gravitational acceleration (both vectors) as *gravity* (the resultant vector). Correspondingly, it is convenient to define a *centrifugal potential* that generates the centrifugal acceleration:

$$\Phi(r,\theta) = \frac{1}{2}\,\omega_e^2\, r^2 \sin^2\theta\,, \tag{3}$$

where ω_e is Earth's rotation rate. The *gravity potential* is then simply

$$W(r,\theta,\lambda) = V(r,\theta,\lambda) + \Phi(r,\theta)\,. \tag{4}$$

A surface on which a potential function has a constant value is called an *equipotential surface*; and, the equipotential surface of W, $W(r,\theta,\lambda) = \text{constant} = W_0$, that closely agrees with mean sea level is known as the *geoid* (first introduced by C.F. Gauss in 1828 to help define the shape of the Earth, later in 1873 clarified by J.B. Listing as specifically associated with the oceans, and in modern views understood to vary in time due to mass deformations and redistribution; see also Grafarend, 1994). The distance between the ellipsoid and the geoid is known as the *geoid undulation*, or also the *geoid height*.

Immediately, we note that equation (1), in principle, cannot be used to compute the potential of the geoid, W_0, in land areas, because the geoid generally lies within the continental crust and the model (1) holds only in free space. In addition, we know that mean sea level and the land (and ocean bottom) surface are affected by the gravitational attractions of the sun and moon (and negligibly by the other planets), which must be recognized in our definition of the geoid. Both of these issues affect our definitions of vertical datum surface (Bursa et al. (1997) show that the potential *value* of the geoid, however, is not affected).

The relationship between the gravity vector and its potential is simply

$$\mathbf{g} = \boldsymbol{\nabla} W , \tag{5}$$

where $\boldsymbol{\nabla}$ denotes gradient operator. Generally, we know from the calculus that the gradient is a vector pointing in the direction of steepest descent of a function, that is, perpendicular to its isometric lines–for example, in the case of the potential, it is the vector perpendicular to the equipotential surfaces. The component of gravity perpendicular to the equipotential surface thus embodies its magnitude, and we may write

$$|\mathbf{g}| = g = -\frac{dW}{dn} , \tag{6}$$

where dn is a differential path along the perpendicular and the minus sign is a matter of convention (potential decreases with altitude, path length is positive upwards, and gravity magnitude is positive).

2.2 Dynamic height

The height of a point "above mean sea level" now is defined more precisely as a height with respect to the geoid, which is a well defined surface, in principle, although its accessibility has yet to be established. In fact, we will first define a *local* geoid (*local vertical datum*) with a single point, $P_0^{(j)}$, that is assumed to be on it and accessible (e.g., a tide gauge station), and where the gravity potential is $W_0^{(j)}$ (not necessarily a known value).

There are three types of geoid-referenced height; each fundamentally refers to the difference in gravity potential between the (local) geoid and the point in question. This potential difference is known as the *geopotential number*:

$$C_P^{(j)} = W_0^{(j)} - W_P , \tag{7}$$

where W_P is the gravity potential at the point, P. Any point has a unique geopotential number (with respect to the defined local geoid), and this, itself, appropriately scaled, can be used as a height coordinate of the point. Specifically, for an adopted constant, γ_0, we have

$$H_P^{dyn(j)} = \frac{C_P^{(j)}}{\gamma_0} \,, \tag{8}$$

which is the *dynamic height* of P (with respect to vertical datum, j). The scale factor is usually chosen as a nominal value of gravity at mid-latitude ($\gamma_0 = 9.806199203\ \mathrm{m/s^2}$; GRS80 value, Moritz, 1992). Noting that the magnitude of gravity is approximately the vertical gradient of the potential (equation (6)), we see that the dynamic height, defined by (8), looks like a height (it has units of distance). On the other hand, in fact, it has no geometric meaning; it is a purely physical quantity–it is the potential (in distance units) relative to the geoid. Clearly, the same constant scale factor must be used for all dynamic heights within a particular datum.

2.3 Orthometric Height

Seeking a geometric definition of height, that is, in terms of an actual vertical distance, we may proceed from the relationship between gravity and potential, (6), integrated as follows:

$$W_P = W_0^{(j)} - \int_{P_0^{(j)}}^{P} g\, dn \,, \tag{9}$$

where the path of integration from the local geoid to P is arbitrary (the gravity field is *conservative*) and could, for example, be an actual path along the Earth's surface. Note, moreover, that the initial point is also arbitrary as long as it is on the datum surface since the local geoid is an equipotential surface (same $W_0^{(j)}$). With (7), we have the intermediate result:

$$C_P^{(j)} = \int_{P_0^{(j)}}^{P} g\, dn \,, \tag{10}$$

showing that geopotential numbers can be determined from measurements of gravity and vertical increments between equipotential surfaces along the path; the latter are obtained with spirit leveling.

Consider now the special path that is always perpendicular to the equipotential surfaces of W. This is called the *plumb line*, and suppose that the path of integration in (10) is along this plumb line. Then

$$C_P^{(j)} = \int_{\bar{P}^{(j)}}^{P} g\, dH\,, \tag{11}$$

where dH is a differential element along the plumb line and $\bar{P}^{(j)}$ is at the base of the plumb line on the local geoid. Dividing and multiplying the right side by the total length of the plumb line, $H_P^{(j)}$, we obtain

$$H_P^{(j)} = \frac{C_P^{(j)}}{\bar{g}_P^{(j)}}\,, \tag{12}$$

where

$$\bar{g}_P^{(j)} = \frac{1}{H_P^{(j)}} \int_{\bar{P}^{(j)}}^{P} g\, dH \tag{13}$$

is the average value of gravity along the plumb line. $H_P^{(j)}$ is known as the *orthometric height* of P (with respect to the defined local vertical datum); it has a very definite geometric interpretation as a distance above the local geoid (along the plumb line, which is curved since the equipotential surfaces are not parallel); see Figure 1.

Unfortunately, while $C_P^{(j)}$ can be measured using (11), the value of $\bar{g}_P^{(j)}$ cannot be *computed* exactly (using Newton's Law of Gravitation) because this would require complete knowledge of the mass density of the crust (and it is not practical to measure g along the plumb line). Therefore, in theory, the orthometric height cannot be determined exactly, and its calculation depends on some density hypothesis, or model, for the crust. A commonly used model assumes constant crustal density and constant topographic height in the vicinity of the point, P. Then, the average gravity along the plumb line, between and $\bar{P}^{(j)}$ and P, is obtained using the *Prey reduction*. This reduction models the value of gravity inside the crust by first removing a Bouguer plate of density, ρ, applying a free-air downward continuation using the normal gradient of gravity, and restoring the Bouguer plate. Each of these operations is linear in H; and, consequently, the average value of gravity along the plumb line is simply the average of the endpoint values of gravity, that is:

$$\bar{g}_P^{\mathrm{Prey}(j)} = \frac{1}{2}\left[g_P + \left(g_P - 2\pi k\rho\, H_P^{(j)} + \frac{\partial\gamma}{\partial h} H_P^{(j)} - 2\pi k\rho\, H_P^{(j)}\right)\right]. \tag{14}$$

With nominal values for the density and gradient (Heiskanen and Moritz, 1967), we obtain

$$\bar{g}_P^{\mathrm{Prey}(j)} = g_P - 2\pi k\rho\, H_P^{(j)} + \frac{1}{2}\frac{\partial\gamma}{\partial h} H_P^{(j)}$$

$$= g_P + (0.0424 \text{ mgal/m})\, H_P^{(j)} . \tag{15}$$

Utilizing (15) in (12) (i.e., $\bar{g}_P^{(j)} \approx \bar{g}_P^{\mathrm{Prey}(j)}$) yields *Helmert heights*:

$$H_P^{\mathrm{Helmert}(j)} = \frac{C_P^{(j)}}{\bar{g}_P^{\mathrm{Prey}(j)}} , \tag{16}$$

which, in principle, requires an iteration on $H_P^{(j)}$; or, since (12) with (15) is just a quadratic in $H_P^{(j)}$, we may write

$$H_P^{\mathrm{Helmert}(j)} = \frac{C_P^{(j)}}{g_P}\left(1 - (0.0424 \text{ mgal/m})\frac{C_P^{(j)}}{g_P^2} + (0.00180 \text{ mgal}^2/\text{m}^2)\left(\frac{C_P^{(j)}}{g_P^2}\right)^2 - \ldots\right), \tag{17}$$

where higher-order terms are less than $O\left(10^{-10}\right)$.

2.4 Normal Height

It is possible to define a similar geometrically interpretable height that avoids a density hypothesis for the crust. This is accomplished by introducing an approximation to the gravity field that can be calculated exactly at any point. The *normal gravity field* suits this purpose. It is defined as the gravity field generated by an Earth-fitting ellipsoid that contains the total mass of the Earth (including the atmosphere), that rotates with the Earth around its minor axis, and that is, itself, an equipotential surface of the gravity field it generates. The gravitational part, V_{ellip}, of the normal field can be expressed as in (1), but because of the imposed symmetries of the ellipsoid and the boundary values, the series contains only even zonal harmonics (no dependence on longitude). The centrifugal part is also given by (3), and the total *normal gravity potential* is

$$U(r,\theta) = V_{\mathrm{ellip}}(r,\theta) + \Phi(r,\theta) . \tag{18}$$

On the ellipsoid, U is a constant, U_0, by definition. U can be calculated anywhere in the space above the ellipsoid using four constants that describe the size and shape of the ellipsoid, it mass, and its rotation. Nowadays, one typically uses:

a = semi–major axis of the ellipsoid ;

J_2 = dynamical flattening (second–degree zonal harmonic coefficient) ; (19)

kM = Newton's gravitational constant times Earth's mass (including atmosphere) ;

ω_e = Earth's rate of rotation .

The value of the potential, U_0 , is completely determined by the adopted constants in (19); it is given by the Pizzetti formula (Heiskanen and Moritz, 1967):

$$U_0 = \frac{kM}{E} \tan^{-1}\frac{E}{b} + \frac{1}{3}\, \omega_e^2\, a^2 \, , \tag{20}$$

where b is the semi-minor axis of the normal ellipsoid, and E it its linear eccentricity. The *normal gravity vector* is, analogous to (5):

$$\boldsymbol{\gamma} = \boldsymbol{\nabla} U \, , \tag{21}$$

and can, likewise, be calculated exactly anywhere on or above the ellipsoid from the given constants (NIMA, 1997).

Consider now the *normal plumb line* through P; it is the line that is always perpendicular to the equipotential surfaces of the normal gravity field. On that line is a point, Q, where the normal gravity potential equals the actual gravity potential at P:

$$U_Q = W_P \, . \tag{22}$$

Note that U and W refer to two distinct gravity fields, and the equality above is not a functional relationship, simply an assignment of values. We define the *normal geopotential number* of Q (also called *spheropotential number*) as follows:

$$C_Q^{\text{normal}} = U_0 - U_Q = U_0 - W_P = C_P^{(j)} + \left(U_0 - W_0^{(j)}\right) , \tag{23}$$

where (7) was used and where, analogous to (11):

$$C_Q^{\text{normal}} = \int_{\bar{Q}}^{Q} \gamma \, dH^* \, , \tag{24}$$

and $\bar{Q}$ is the base point on the ellipsoid (not the geoid!) of the normal plumb line (see Figure 2). Also dH^* denotes a differential path element along the normal plumb line.

Dividing and multiplying the right side of (24) by the total length, H_Q^*, of the normal plumb line from the ellipsoid to Q, we obtain with (23):

$$H_Q^* = \frac{C_P^{(j)} + \left(U_0 - W_0^{(j)}\right)}{\bar{\gamma}_Q}, \tag{25}$$

where

$$\bar{\gamma}_Q = \frac{1}{H_Q^*} \int_{\bar{Q}}^{Q} \gamma \, dH^* \tag{26}$$

is the average value of normal gravity along the normal plumb line. Now, formally, the point Q lies on the *telluroid* and the distance between the telluroid and the Earth's surface is known as the *height anomaly* at P, ζ_P. Conventionally, the distances H_Q^* and ζ_P are "reversed" along the normal plumb line, and the surface defined by the separation ζ_P from the ellipsoid is known as the *quasi-geoid*; see Figure 2. The shape of the quasi-geoid is similar to that of the geoid, but the quasi-geoid is not an equipotential surface in either the normal or the actual gravity field. If the point, P, is on the geoid (as it is approximately in ocean areas) and if the gravity potential value of the geoid is $W_0 = U_0$, then the telluroid point, Q, is on the ellipsoid. Thus the quasi-geoid equals the geoid at these points.

We assume that the difference, $U_0 - W_0^{(j)}$, in (25) is generally not known (which is the case, at least historically). Consider the case when P is the origin point, $P_0^{(j)}$, of the local vertical datum. Then $C_{P_0^{(j)}}^{(j)} = 0$ and for the corresponding telluroid point, $Q_0^{(j)}$, we have

$$H_{Q_0^{(j)}}^* = \frac{\left(U_0 - W_0^{(j)}\right)}{\bar{\gamma}_{Q_0^{(j)}}}. \tag{27}$$

$H_{Q_0^{(j)}}^*$ is the distance from the ellipsoid to that point, $Q_0^{(j)}$, where $U_{Q_0^{(j)}} = W_0^{(j)}$, or, alternatively, the distance from the point, $P_0^{(j)}$, to the quasi-geoid.

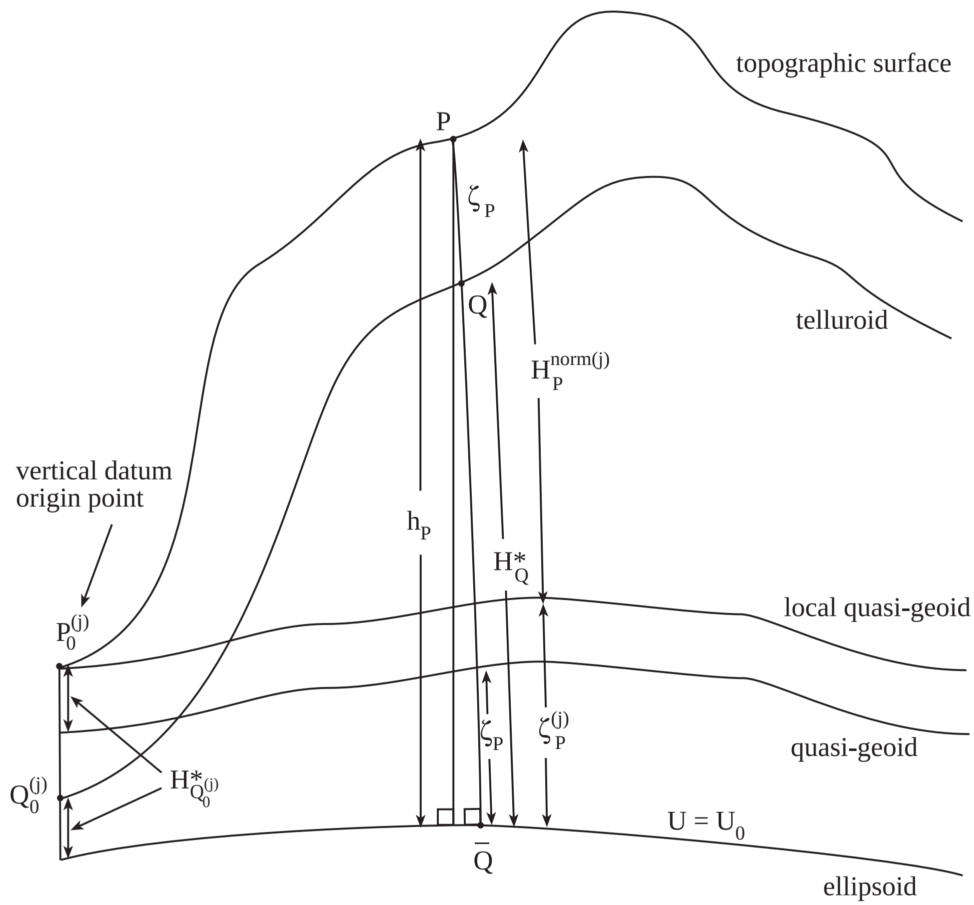

Figure 2: Normal height and height anomaly; also quasi-geoid versus local quasi-geoid.

We now define a *local quasi-geoid* that contains the datum point $P_0^{(j)}$ and is parallel to the quasi-geoid by the amount $H^*_{Q_0^{(j)}}$ (Figure 2). The distance from the local quasi-geoid to the point P is then given the name, *normal height*, and is expressed by (25) relative to $H^*_{Q_0^{(j)}}$, and now with new notation:

$$H_P^{norm(j)} = \frac{C_P^{(j)}}{\bar{\gamma}_Q} . \tag{28}$$

Using an approximate form of normal gravity as a function of (ϕ,h), Heiskanen and Moritz (1967) derive the average normal gravity, $\bar{\gamma}_Q$, using (26):

$$\bar{\gamma}_Q \approx \gamma \left[1 - \left(1 + f + \frac{\omega_e^2 a^2 b}{kM} - 2f \sin^2\phi \right) \frac{H_P^{norm(j)}}{a} + \left(\frac{H_P^{norm(j)}}{a} \right)^2 \right], \tag{29}$$

and, substituting this into (28) and inverting, they find an expression for the normal height:

$$H_P^{norm(j)} \approx \frac{C_P^{(j)}}{\gamma}\left[1 + \left(1 + f + \frac{\omega_e^2 a^2 b}{kM} - 2f\sin^2\phi\right)\frac{C_P^{(j)}}{a\gamma} + \left(\frac{C_P^{(j)}}{a\gamma}\right)^2\right]. \qquad (30)$$

We note for future reference that

$$H_P^{norm(j)} = H_Q^* - H_{Q_0^{(j)}}^* . \qquad (31)$$

Also, we can define the *local height anomaly*, identifying the separation of the local quasi-geoid from the ellipsoid (Figure 2):

$$\zeta_P^{(j)} = \zeta_P + H_{Q_0^{(j)}}^* = \zeta_P + \frac{\left(U_0 - W_0^{(j)}\right)}{\gamma_{Q_0^{(j)}}} . \qquad (32)$$

2.5 Review of Heights

It is important to realize that the normal height depends in the first place, like the dynamic and orthometric heights, on the geopotential number at P, $C_P^{(j)}$ (which is the actual geopotential value with respect to the local geoid potential). Unlike orthometric heights, normal heights can be determined exactly, albeit some iterative procedure is required in the computation of $\bar{\gamma}_Q$, since it depends also on the normal height. And, unlike dynamic heights, normal heights have a definite geometric interpretation, as the vertical distance of P above the local quasi-geoid. Finally, we will see that geopotential models, such as (1), yield the height anomaly more readily than the geoid undulation (in fact, the latter, in theory, is not determinable because we do not know the crustal mass density); and, therefore, the quasi-geoid is theoretically more realizable from geopotential models than the geoid. One disadvantage of the normal height is its arcane definition, being the height above the quasi-geoid and not the geoid (which is usually, though with some error, identified as mean sea level).

A disadvantage of both orthometric and normal heights is that neither indicates the direction of flow of water. Only dynamic heights possess this property. That is, two points with identical dynamic heights are on the same equipotential surface (of the actual gravity field) and water will not flow from one to the other point. Two points with identical orthometric heights lie on different equipotential surfaces (since, generally, $\bar{g}_{P_1} \neq \bar{g}_{P_2}$); and, water *will* flow from one point to the other, even though they have the same (orthometric) height. The same statement holds for normal heights, although, because of the smoothness of the normal field, the effect is not as severe.

Table 1 gives a summary of the three types of heights, as well as variations of these based on different assumptions and approximations. The reader is cautioned that the nomenclature of the approximate forms and the approximations, themselves, are not universal and can lead to confusion. The given formulas correspond to definitions given in NGS (1986).

Table 1: Height definitions according to NGS (1986).

ellipsoidal height (geodetic height)	h_P	$h_P = h_P\left(x_P, y_P, z_P, a, f\right)$
dynamic height	$H_P^{dyn(j)} = \dfrac{C_P^{(j)}}{\gamma_0}$	$\gamma_0 = \gamma\left(\phi_0\right)$, $\phi_0 = 45°$
normal dynamic height		$C_P^{(j)} \approx \int_{P_0^{(j)}}^{P} \gamma\, dH^*$, $\gamma_0 = \gamma\left(\phi_0\right)$
orthometric height	$H_P^{(j)} = \dfrac{C_P^{(j)}}{\bar{g}_P^{(j)}}$	$\bar{g}_P^{(j)} = \dfrac{1}{H_P^{(j)}} \int_{\bar{P}^{(j)}}^{P} g\, dH$
Helmert height		$\bar{g}_P^{(j)} \approx g_P - 2\pi k\rho\, H_P^{(j)} + \dfrac{1}{2} \dfrac{\partial\gamma}{\partial h} H_P^{(j)}$
Niethammer height		$\bar{g}_P^{(j)} \approx \bar{g}_P^{Prey(j)} + \delta g_t$ †
normal orthometric height		$C_P^{(j)} \approx \int_{P_0^{(j)}}^{P} \gamma\, dH$, $\bar{g}_P^{(j)} \approx \gamma\left(\phi_P, \tfrac{1}{2} H_P^{(j)}\right)$
normal height (Molodensky height)	$H_P^{norm(j)} = \dfrac{C_P^{(j)}}{\bar{\gamma}_Q}$	$\bar{\gamma}_Q = \dfrac{1}{H_Q^*} \int_{\bar{Q}}^{Q} \gamma\, dH^*$
quasi-dynamic height		$\bar{\gamma}_Q \approx \gamma\left(\phi_0\right) + \dfrac{H_P^{(j)}}{2} \left.\dfrac{\partial\gamma}{\partial h}\right\vert_{h = H_P^{(j)}}$
height anomaly	$\zeta_P^{(j)}$	$\zeta_P^{(j)} = \dfrac{T_P}{\gamma_Q} + \dfrac{\left(U_0 - W_0^{(j)}\right)}{\gamma_{Q_0^{(j)}}}$
geoid height (geoid undulation)	$N_P^{(j)}$	$N_P^{(j)} = \dfrac{T_{\bar{P}^{(j)}}}{\gamma_{Q^{(j)}}} + \dfrac{1}{\gamma_{\bar{Q}}} \left(U_{\bar{Q}} - W_0^{(j)}\right)$

† δg_t accounts for the terrain relative to the Bouguer plate in a modified Prey reduction.

Clearly, given sufficient gravity information, the dynamic, orthometric, and normal heights can be transformed from one to the other, because they all depend firstly on the geopotential number.

From (8), (12), and (28), we have

$$C_P^{(j)} = \gamma_0 H_P^{\mathrm{dyn}(j)} = \bar{g}_P^{(j)} H_P^{(j)} = \bar{\gamma}_Q H_P^{\mathrm{norm}(j)} . \tag{33}$$

The difference between a length along the plumb line and a corresponding length along the perpendicular (or normal) to the ellipsoid is due to the curvature of the former. The curvature results in a deflection of the plumb line from the ellipsoid normal (deflection of the vertical, Θ) that is usually of the order of 10 arcsec and in rare cases can reach 1 arcmin. With reference to Figure 3, we see that the corresponding height difference is

$$\delta h \approx h \sin\Theta \tan\Theta . \tag{34}$$

This is a negligible effect for all topographic heights of the Earth (even for the extreme case of $\Theta = 1$ arcmin and $h = 10000$ m, we obtain $\delta h < 1$ mm). Thus we can treat all geometrically interpretable heights as lengths along the ellipsoidal normal, which considerably simplifies comparisons and conversions among the different heights.

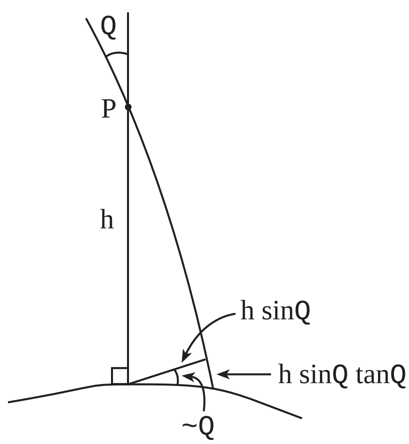

Figure 3: The difference between lengths along the curved plumb line and along the straight ellipsoid perpendicular.

3. Models for $W_0^{(j)}$

We wish to find a means to determine the potential value of the local vertical datum, j. Then we are able to relate different datums around the world and also define a world vertical datum. Clearly, if we have an estimate of the geopotential (e.g., EGM96, Lemoine et al., 1998) and we know the (geocentric) coordinates of a point on the surface of the local vertical datum (the local geoid), then it is simply a matter of evaluating the gravity potential at this point to find $W_0^{(j)}$. The problem generalizes if the point is not on the local geoid, but we know its height with respect to the datum and thus, again, have the requisite coordinates to evaluate the gravity potential. Thus, overall, we require the following data to make an estimation of $W_0^{(j)}$: an estimate of the gravity potential function, W; the geocentric coordinates of a point (e.g., r,θ,λ), and the height of this point with respect to the datum.

We start with an expansion of the normal gravity potential in a Taylor series along the ellipsoid normal:

$$U_P = U_Q + \left(h_P - h_Q\right) \left.\frac{\partial U}{\partial h}\right|_{h=h_Q} + \frac{1}{2!}\left(h_P - h_Q\right)^2 \left.\frac{\partial^2 U}{\partial h^2}\right|_{h=h_Q} + \cdots . \tag{35}$$

Now, with

$$\gamma_Q = -\left.\frac{\partial U}{\partial h}\right|_{h=h_Q} \tag{36}$$

and with (22), we obtain:

$$h_P - h_Q = \frac{1}{\gamma_Q}\left(W_P - U_P\right) - \frac{1}{2\,\gamma_Q}\left(h_P - h_Q\right)^2 \left.\frac{\partial \gamma}{\partial h}\right|_{h=h_Q} + \cdots . \tag{37}$$

The left side is the height anomaly, ζ_P. The second term (and higher-order terms) on right side can be ignored; since $\left|h_P - h_Q\right| < 110\text{ m}$ and the vertical gradient of gravity is 0.3086 mgal/m, it amounts to no more than 2 mm. Defining the *disturbing potential*, T_P, at point P:

$$T_P = W_P - U_P , \tag{38}$$

we then obtain from (37) the height anomaly in *physical* rather than geometric terms:

$$\zeta_P = \frac{T_P}{\gamma_Q} . \tag{39}$$

From (32), we obtain the local height anomaly, also in terms of the disturbing potential:

$$\zeta_P^{(j)} = \frac{T_P}{\gamma_Q} + \frac{\left(U_0 - W_0^{(j)}\right)}{\gamma_{Q_0^{(j)}}} . \tag{40}$$

If the point, P, is on the local geoid ($P = \bar{P}^{(j)}$), then we obtain similarly as in (37):

$$h_{\bar{P}^{(j)}} - h_{Q^{(j)}} = \frac{T_{\bar{P}^{(j)}}}{\gamma_{Q^{(j)}}}, \tag{41}$$

where $Q^{(j)}$ is the point at which $U_{Q^{(j)}} = W_{\bar{P}^{(j)}}$. The *local geoid undulation* is obtained as follows. First note that by again applying (35) ($P \to Q^{(j)}$ and $Q \to \bar{Q}$), we have

$$\begin{aligned} h_{Q^{(j)}} - h_{\bar{Q}} &= \frac{1}{\gamma_{\bar{Q}}}\left(U_{\bar{Q}} - U_{Q^{(j)}}\right) \\ &= \frac{1}{\gamma_{\bar{Q}}}\left(U_{\bar{Q}} - W_{\bar{P}^{(j)}}\right). \end{aligned} \tag{42}$$

Thus, with the geoid undulation, $N_P^{(j)} = h_{\bar{P}^{(j)}} - h_{\bar{Q}}$, and $W_{\bar{P}^{(j)}} = W_0^{(j)}$ we obtain

$$N_P^{(j)} = \frac{T_{\bar{P}^{(j)}}}{\gamma_{Q^{(j)}}} + \frac{1}{\gamma_{\bar{Q}}}\left(U_{\bar{Q}} - W_0^{(j)}\right), \tag{43}$$

Equations (39), (40), and (43) are manifestations of the (*generalized*) *Bruns' formula*.

From leveling (and gravity data) we obtain the normal height, $H_P^{\mathrm{norm}(j)}$, with respect to the local quasi-geoid according to (11), (26), and (28). From GPS we obtain the ellipsoidal height, h_P; which allows us to compute at P the actual potential from a model (e.g., EGM96), assumed errorless for the moment. Also, the normal potential can be computed at P, hence T_P can be computed according to (38). From Figure 2, we have

$$h_P = H_P^{\mathrm{norm}(j)} + \zeta_P^{(j)}. \tag{44}$$

Using the expression (40) for the local height anomaly, we can solve for the local geoid potential:

$$W_0^{(j)} = U_0 - \gamma_{Q_0^{(j)}}\left(h_P - H_P^{\mathrm{norm}(j)} - \frac{T_P}{\gamma_Q}\right). \tag{45}$$

All quantities on the right side are either given or measured, and the left side is the actual potential of the local geoid. To verify this equation, select the point, P, to be the origin point, $P_0^{(j)}$, of the local vertical datum; then (45) reduces to

$$W_0^{(j)} = U_0 - \gamma_{Q_0^{(j)}} \left(h_{P_0^{(j)}} - \frac{W_{P_0^{(j)}} - U_{P_0^{(j)}}}{\gamma_{Q_0^{(j)}}} \right)$$

$$= W_{P_0^{(j)}} + U_0 - U_{P_0^{(j)}} - \gamma_{Q_0^{(j)}} h_{P_0^{(j)}} \tag{46}$$

$$= W_{P_0^{(j)}} ,$$

as it should be (to first-order approximation).

Re-defining the unknown parameter as

$$\Delta H_0^{(j)} = \frac{U_0 - W_0^{(j)}}{\gamma_{Q_0^{(j)}}} , \tag{47}$$

we have the following model

$$\Delta H_0^{(j)} = h_P - H_P^{\text{norm}(j)} - \zeta_P . \tag{48}$$

A similar model can be derived that involves the orthometric height. In this case, we must assume that the gravity potential model can be evaluated at the local geoid point, $\bar{P}^{(j)}$ (the coordinates of $\bar{P}^{(j)}$ come from the ellipsoidal height obtained by GPS and the orthometric height, $H_P^{(j)}$, obtained by leveling and gravity (and a density assumption); and we still need a density assumption to compute the potential inside the mass). Then from Figure 2, we have

$$h_P = H_P^{(j)} + N_P^{(j)} ; \tag{49}$$

and, if we substitute (43), we again obtain an equation for the local geoid potential:

$$W_0^{(j)} = U_{\bar{Q}} - \gamma_{\bar{Q}} \left(h_P - H_P^{(j)} - \frac{T_{\bar{P}^{(j)}}}{\gamma_{Q^{(j)}}} \right) . \tag{50}$$

Because of the density hypotheses needed to compute both the orthometric height and the disturbing potential, this model is less useful for precise determinations of the local geoid potential. If orthometric heights are available at certain points of the datum, then it would be advantageous to first convert these to normal heights, using (33), and then apply the model (48).

This section concludes with formulas for the orthometric and normal heights of a particular local vertical datum, determined on the basis of the gravity potential and ellipsoidal heights. Often, nowadays, the determination of the geoid undulation (or, more correctly, the gravity potential function) is justified on the basis that it provides orthometric heights (or, normal heights) if ellipsoidal heights are given, the latter being readily determined by satellite methods such as GPS. Clearly, equations (44) and (49) are the basis for these determinations. Note, however, that it is the geoid undulation for the local geoid, or the local vertical datum, that is required, which implies that in addition to the gravity potential function, W (or T), the potential value of the local geoid must be known. That is, substituting (43) into (49) yields

$$H_P^{(j)} = h_P - \frac{T_{\bar{P}^{(j)}}}{\gamma_{Q^{(j)}}} - \frac{1}{\gamma_{\bar{Q}}}\left(U_{\bar{Q}} - W_0^{(j)}\right). \tag{51}$$

Similarly, for the normal height with respect to the local vertical datum, we substitute (40) into (44) to obtain

$$H_P^{norm(j)} = h_P - \frac{T_P}{\gamma_Q} - \frac{1}{\gamma_{Q_0^{(j)}}}\left(U_0 - W_0^{(j)}\right). \tag{52}$$

Thus, we see the importance of knowing the local geoid potential, $W_0^{(j)}$, in order to determine orthometric (or normal) heights in a local vertical datum with GPS and the disturbing potential function.

4. References

Bursa, M., K. Radej, Z. Sima, S.A. True, and V. Vatrt (1997): Determination of the geopotential scale factor from TOPEX/Poseidon satellite altimetry. *Studia geoph. et geod.*, **41**, 203-216.

Grafarend, E.W. (1994): What is a geoid? In: *Geoid and its Geophysical Interpretation*, P. Vanicek and N.T. Christou (eds.), pp.1-32, CRC Press, Boca Raton.

Lemoine, F.G. et al. (1998): The development of the joint NASA GSFC and the National Imagery Mapping Agency (NIMA) geopotential model EGM96. NASA Technical Report NASA/TP-1998-206861, Goddard Space Flight Center, Greenbelt, Maryland.

Heiskanen, W.A. and H. Moritz (1967): *Physical Geodesy*, Freeman and Co., San Francisco.

Moritz, H. (1992): Geodetic Reference System 1980. *Bulletin Géodésique*, **66**(2), 187-192.

NGS (1986): *Geodetic Glossary*. Publication of the National Geodetic Survey, NOAA/NOS, National Geodetic Information Center, Rockville, MD.

NIMA (1997): Department of Defense World Geodetic System 1984. Tech. Report TR-8350.2, third edition, National Imagery and Mapping Agency, Bethesda, MD.

Sjöberg, L.E. (1999): The IAG approach to the atmospheric geoid correction in Stokes' formula and a new strategy. *Journal of Geodesy*, **73**(7), 362-366.