Ellipsoidal Geodetic Coordinate Systems

— A Primer —

Jürgen Kusche
Department of Geodesy, Delft University of Technology

j.kusche@citg.tudelft.nl

Lecture Notes to ge2211

— preliminary version —

last revision: September 25, 2001

Preface

This primer is meant as an introduction into the basic ellipsoidal coordinate systems applied in geodesy, navigation, geo–referencing and geo–information technology. Common examples for such coordinate systems are transverse Mercator–coordinates (world–wide in use as UTM–system), stereographic coordinates (like the dutch RD–coordinates or the polar UPS–system), and, of course, the familiar geodetic λ, ϕ–coordinates.

In these fields we are dealing with two–dimensional curvilinear coordinates. *Curvilinear* means that the coordinate systems are defined on (parts of) a curved surface of reference. In the applications mentioned above, this surface of reference is chosen routinely as an ellipsoid of revolution. As a consequence, coordinate lines are in general never straight lines and in most cases not even representable by closed expressions. *Two–dimensional* means that we refer to positions on the reference surface by attaching a pair of surface coordinates to them. It is straightforward then to define three–dimensional coordinates simply by adding a "height" to the surface coordinates, which itself is measured with respect to the same reference surface (as with the ellipsoidal heights) or to a different reference surface (the geoid, for example).

The focus here is on the differential geometric background and on the ideas and concepts, which lead to the definition of so much different surface coordinate systems. Also some methods to derive the transformation equations – preferably by series expansions – between different systems are covered. We try to explain things first in general, i. e. valid for arbitrary surfaces, and specialize afterwards to the ellipsoid of revolution. More implementation–specific details can be found elsewhere. Free software for coordinate transformation (source codes, Java applets, etc.) is widely available on the internet meanwhile.

Transformation between curvilinear coordinates is often (and particularly in cartography) considered as a *mapping* process, and the transformation equations are interpreted as mapping formulas or projections. For a spherical reference surface this allows geometric representations: Spherical Transverse Mercator coordinates, for instance, may be imagined as derived from a conformal projection of the sphere onto an prescribing horizontally aligned cylinder. Such mapping formulas are often available as closed–form expressions. For an ellipsoid of revolution these intuitive geometric representations usually break down, and for geodetic purposes the closed spherical relations are by far not sufficient. This is the reason why we avoid such geometrically oriented interpretations and arguments in this primer.

Jürgen Kusche — Delft, July 2001

Recommended reading includes

1. Strang van Hees, G., *Globale en lokale geodetische systemen.* Nederlandse Commissie voor Geodesie, Publikatie 30, 1994

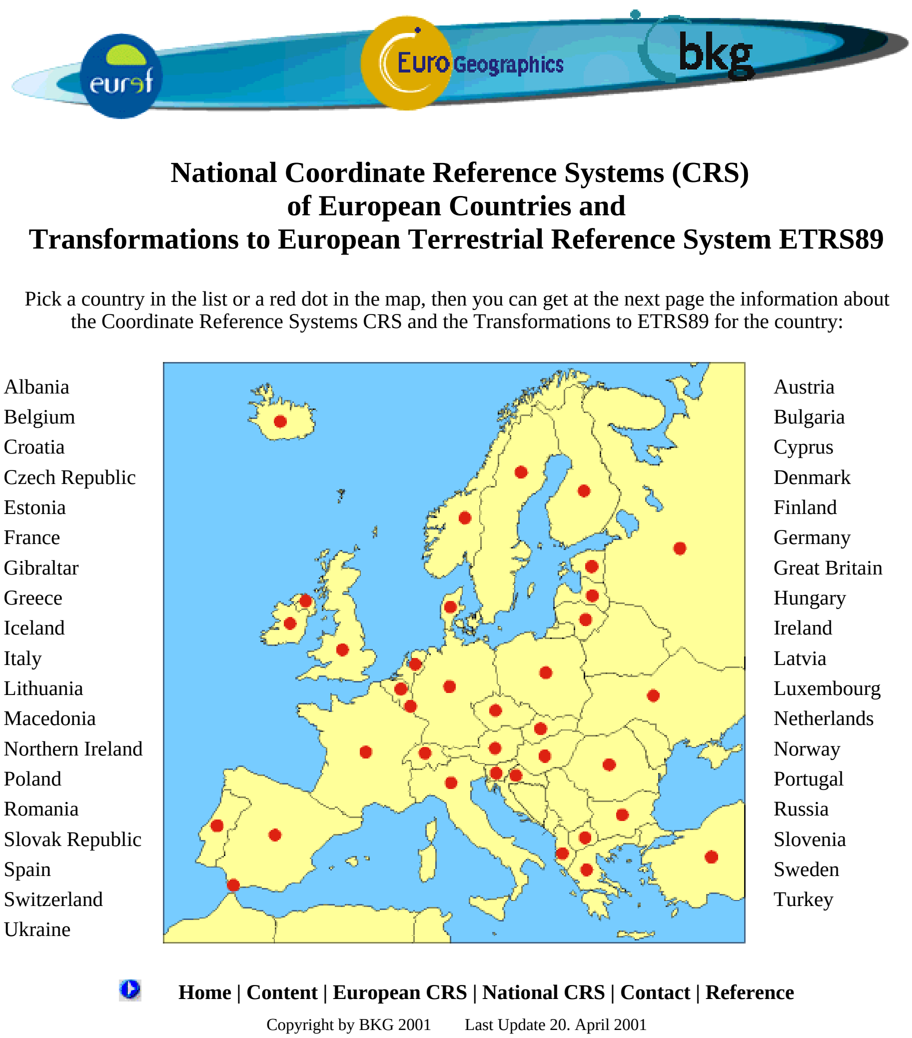
2. Heitz, S., *Coordinates in Geodesy.* Springer, 1988

Technical information can be found, for example, in

1. Defense Mapping Agency, *The universal grids: Universal Transverse Mercator and Universal Polar Stereographic*, Technical Manual, DMATM 8358.2

Further references can be found in the above mentioned publications.

A lot of material is available on the world wide web, do a search with keywords 'mapping', 'UTM', or 'coordinates'!



euref EuroGeographics bkg

National Coordinate Reference Systems (CRS)
of European Countries and
Transformations to European Terrestrial Reference System ETRS89

Pick a country in the list or a red dot in the map, then you can get at the next page the information about the Coordinate Reference Systems CRS and the Transformations to ETRS89 for the country:

Albania	Austria
Belgium	Bulgaria
Croatia	Cyprus
Czech Republic	Denmark
Estonia	Finland
France	Germany
Gibraltar	Great Britain
Greece	Hungary
Iceland	Ireland
Italy	Latvia
Lithuania	Luxembourg
Macedonia	Netherlands
Northern Ireland	Norway
Poland	Portugal
Romania	Russia
Slovak Republic	Slovenia
Spain	Sweden
Switzerland	Turkey
Ukraine	

Home | Content | European CRS | National CRS | Contact | Reference

Copyright by BKG 2001 Last Update 20. April 2001

Homepage of European Coordinate Reference Systems *information service (BKG, Frankfurt,* `http://crs.ifag.de`

Contents

Chapter 1

Differential Geometry on 2D–Surfaces

1.1 Basics of differential geometry

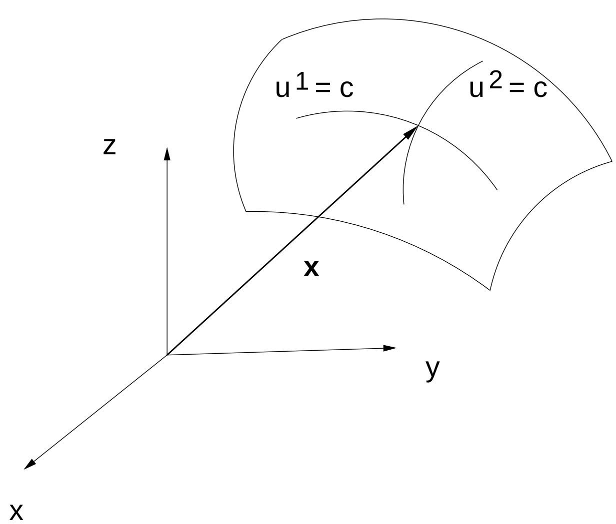
The most simplest and straightforward way to represent mathematically a 2–dimensional curved surface in 3–dimensional space is the *Gaussian representation*. Here, the vector $\boldsymbol{x}$, pointing from the origin of an 3–dimensional x, y, z–coordinate system to an arbitrary point on the surface, is given as a function of two independant parameters u^1 and u^2. The parameters are usually denoted to as Gaussian parameters or *surface coordinates*.

$$\boxed{\boldsymbol{x} = \begin{pmatrix} x(u^1, u^2) \\ y(u^1, u^2) \\ z(u^1, u^2) \end{pmatrix} = \boldsymbol{x}(u^1, u^2) = \boldsymbol{x}(u^\alpha)} \tag{1.1}$$

It is a common convention to make use of *upper* (superscript) Greek indices for 2–dimensional curvilinear coordinates — so u^2 does *not* mean "square of u".

$$\alpha \in \{1, 2\}$$

Thus, Greek indices $\alpha, \beta, \gamma, \ldots$ always take the values 1 or 2. However, in old–style textbooks the coordinates are sometimes denoted by u and v.



Coordinate Lines

If we keep *one* of the coordinates u^1 or u^2 at a given value $c =$ const and consider the other coordinate as an independant parameter, we will obtain a (1–dimensional) space curve. These particular space curves – which are obviously embedded in the surface under consideration – are called *coordinate lines*:

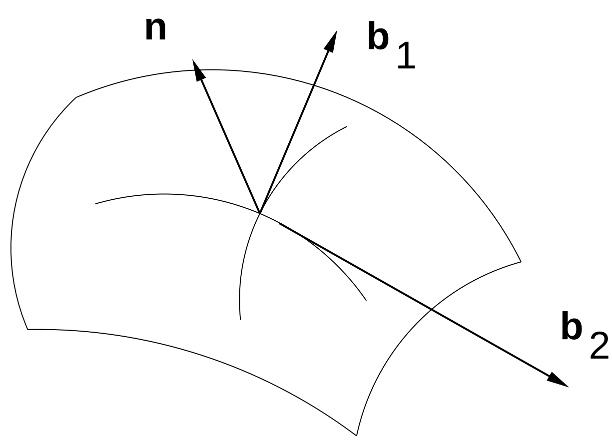
$$u^1 = c = \text{const} \;:\quad \boldsymbol{x}(\mathrm{c}, \mathrm{u}^2) = \boldsymbol{x}(\mathrm{u}^2)$$

$$u^2 = c = \text{const} \;:\quad \boldsymbol{x}(\mathrm{u}^1, \mathrm{c}) = \boldsymbol{x}(\mathrm{u}^1)$$

Next we will look at the *local basis*, which is nothing else than a kind of 'natural' basis of the tangent plane. Consider the vectors

$$\boldsymbol{b}_1 := \frac{\partial \boldsymbol{x}}{\partial u^1} \qquad \boldsymbol{b}_2 := \frac{\partial \boldsymbol{x}}{\partial u^2} \tag{1.2}$$

They are tangent to the coordinate lines, but in general neither perpendicular nor of unit length (since u^1 or u^2 do not necessarily equal to the arc length s). Both $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are contained in the tangent plane, and can be used as basis vectors in order to represent arbitrary vectors of the tangent plane. This means, if $\boldsymbol{a}$ lies in the tangent plane, we may write $\boldsymbol{a} = a^1 \cdot \boldsymbol{b}_1 + a^2 \cdot \boldsymbol{b}_2$ with certain real numbers a^1, a^2. The basis is called *local* since the tangent plane of a curved surface differs from point to point. Of course, there are other possible systems of base vectors for the tangent plane than the $\boldsymbol{b}_\alpha$ defined by (1.2), but these are most straightforward to derive if the Gaussian representation of the surface is given.



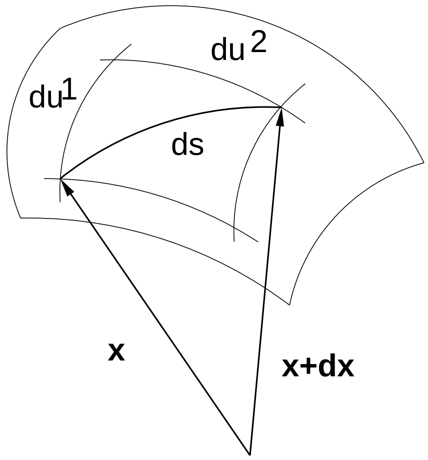
Local basis

Since both basis vectors are contained in the tangent plane, the unit normal vector $\boldsymbol{n}$ of the surface can be obtained from

$$\boldsymbol{n} = \frac{\boldsymbol{b}_1 \times \boldsymbol{b}_2}{||\boldsymbol{b}_1 \times \boldsymbol{b}_2||}$$

An important quantity in differential geometry is the *arc length s* along arbitrary curves on the surface. It can be (at least in theory) measured directly, using a tape for example or by counting your steps when "walking" on the surface (imagine an ant crawling on the surface of an apple). Therefore it must be *invariant* against the choice of the coordinate system as well as against rotations and translations of the surface as a whole in 3–dimensional space.

The differential ds of the arc length (thus the length of an infitesimal short arc) is called the *line element.* It is interesting to look closer at the line element since for small (in the limit infitesimal small) regions every curved surface may be approximated ((in the limit, replaced) by its tangent plane, regardless "how strong" the curvature is.



Line element

For the line element ds, connecting two (infinitely close) neighbouring points $\boldsymbol{x} = \boldsymbol{x}(u^\alpha)$ and $\boldsymbol{x'} = \boldsymbol{x}'(u^\alpha + du^\alpha) = \boldsymbol{x} + \boldsymbol{dx}$, one has

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \boldsymbol{dx} \cdot \boldsymbol{dx} \\ &= \left(\frac{\partial \boldsymbol{x}}{\partial u^1} du^1 + \frac{\partial \boldsymbol{x}}{\partial u^2} du^2 \right) \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u^1} du^1 + \frac{\partial \boldsymbol{x}}{\partial u^2} du^2 \right) \\ &= \left(\boldsymbol{b}_1 \, du^1 + \boldsymbol{b}_2 \, du^2 \right) \cdot \left(\boldsymbol{b}_1 \, du^1 + \boldsymbol{b}_2 \, du^2 \right) \end{aligned}$$

Here we made use of the fact that for infinitely small domains the distance along the curved surface equals to the distance measured in 3–dimensional space. This can be written

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2$$

when collecting all scalar products of the base vectors in the *metric tensor*

$$g_{\alpha\beta} := \boldsymbol{b}_\alpha \cdot \boldsymbol{b}_\beta = \begin{pmatrix} \boldsymbol{b}_1 \cdot \boldsymbol{b}_1 & \boldsymbol{b}_1 \cdot \boldsymbol{b}_2 \\ \boldsymbol{b}_2 \cdot \boldsymbol{b}_1 & \boldsymbol{b}_2 \cdot \boldsymbol{b}_2 \end{pmatrix}$$

What is a *tensor*? The answer is quite involved, but for the moment we can say that it is an object which can be written as a matrix but is invariant against coordinate transformations, just like a vector. This means that the object as a whole is invariant, not its individual components.

Finally we may write the line element (in the most compressed and very elegant notation)

$$\boxed{ds^2 = g_{\alpha\beta}\; du^\alpha\; du^\beta} \tag{1.3}$$

Here we applied *Einstein's summation convention*: Whenever a Greek index appears *twice* in a product, this means, that we *sum up* over 1 and 2. The summation signs are simply omitted, which will save a lot of writing in what follows.

$$ds^2 = g_{\alpha\beta}\; du^\alpha\; du^\beta = \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\; g_{\alpha\beta}\; du^\alpha\; du^\beta$$

Eq. (1.3) is known as the *1st fundamental form* of the surface. It is a quadratic form, and in matrix–vector notation it may be finally written as $ds^2 = \boldsymbol{v}^T\boldsymbol{G}\boldsymbol{v}$, with $\boldsymbol{v} = (du^1, du^2)^T$. It relates the line element, a measurable quantity (remember an ant crawling on an apple's surface and counting steps), to coordinate differentials. This means, when we know the metric tensor of a surface and the coordinate difference of two (infinitesimal closely) neighbouring points, we can derive the (infinitesimal) small distance between them. The 1st fundamental form is therefore the generalization of Pythagoras' rule $\Delta s^2 = \Delta x^2 + \Delta y^2$ (which is valid only for cartesian coordinates in the plane) for arbitrary coordinates and curved surfaces. However, unlike Pythagoras' rule, the first fundamental form can be applied only to (infinitesimal) small areas.

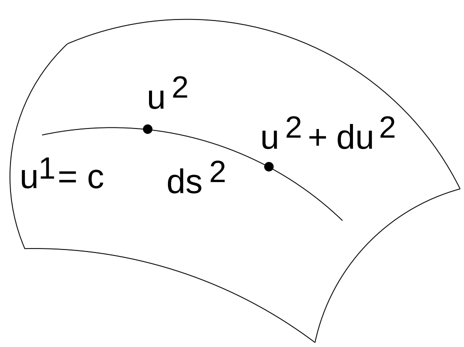
Some notes from a practical viewpoint: When the Gaussian representation of a surface is given, it is rather straightforward to derive the base vectors and then to form the scalar products to get the metric tensor. Since the base vectors depend on the position on the surface *and* on the choice of the surface coordinates, the same holds for the metric tensor: its 3 independant entries (note that $g_{12} = g_{21}$) are functions of the position

$$g_{\alpha\beta} = g_{\alpha\beta}(u^1, u^2)$$

Along the coordinate lines, $\sqrt{g_{11}}$ and $\sqrt{g_{22}}$ act as scale factors,

$$ds^1 = \sqrt{g_{11}}\; du^1 \qquad ds^2 = \sqrt{g_{22}}\; du^2$$

since either $u^1 = \text{const} \Rightarrow \text{du}^1 = 0$ or $u^2 = \text{const} \Rightarrow \text{du}^2 = 0$. Here ds^1 means a line element along the $u^2 = \text{const}$ line, and ds^2 a line element along the $u^1 = \text{const}$ line.



Line element along coordinate line

If $g_{12} = 0$, the coordinate grid is orthogonal, since this means that for the base vectors (which are tangent to the coordinate lines)

$$\boldsymbol{b}_1 \cdot \boldsymbol{b}_2 = 0$$

holds.

Another invariant property is the length of a vector. Assume the vector $\boldsymbol{a}$ contained in the tangent plane. The we may write $\boldsymbol{a} = a^1 \cdot \boldsymbol{b}_1 + a^2 \cdot \boldsymbol{b}_2$, with components a^1, a^2 with respect to the local basis. The length of the vector $\boldsymbol{a}$, usually expressed by its cartesian components

$$||\boldsymbol{a}||^2 = a_x^2 + a_y^2 + a_z^2 = (a_x \; a_y \; a_z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

can be related to the local basis components by

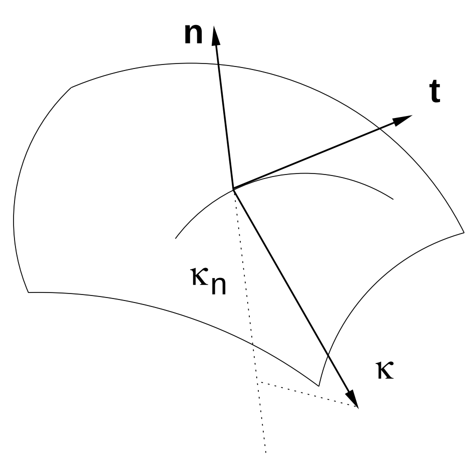
$$||\boldsymbol{a}||^2 = \boldsymbol{a} \cdot \boldsymbol{a} = \left(a^1 \; \boldsymbol{b}_1 + a^2 \; \boldsymbol{b}_2\right) \cdot \left(a^1 \; \boldsymbol{b}_1 + a^2 \; \boldsymbol{b}_2\right) = a^\alpha \; a^\beta \; g_{\alpha\beta}$$

Again, the metric tensor plays the same role than the unit matrix (more precisely the *delta tensor*) in cartesian coordinates.

Finally we mention that in ancient textbooks on differential geometry, g_{11}, g_{12}, g_{22} are sometimes denoted by E, F, G, to the honour of C. F. Gauss.

Apart from the 1st fundamental form, there is also a *2nd fundamental form*, which refers to the *normal curvature* of a surface with respect to a certain direction. It might be less important from our current point of view (which is towards the definition of suitable coordinate systems), but it represents an interesting concept of differential geometry. We consider a surface and an arbitrary curve completely located within the surface: a surface curve. The *normal curvature* κ_n is (by definition) the projection of the total curvature vector $\boldsymbol{\kappa} = d^2\boldsymbol{x}/ds^2$ of the (surface) curve onto the surface normal vector,

$$\kappa_n = \frac{d^2\boldsymbol{x}}{ds^2} \cdot \boldsymbol{n}$$



Normal vector $\boldsymbol{n}$*, tangent vector* $\boldsymbol{t}$*, and total curvature vector* $\boldsymbol{\kappa}$

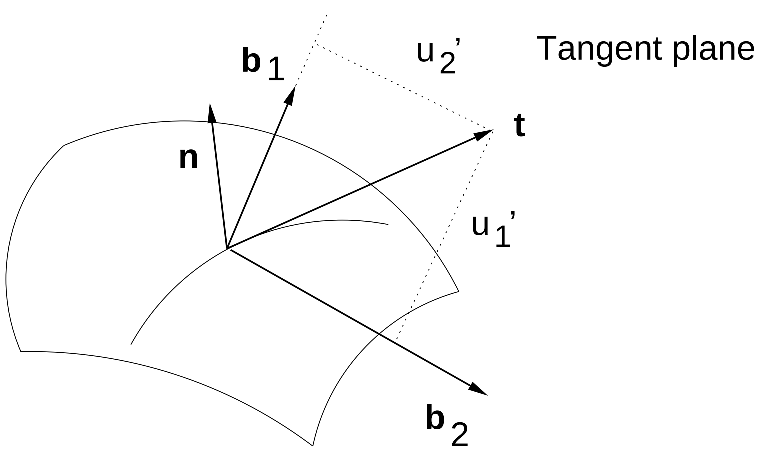
We will understand the meaning of this definition from the following. First, remember that the curvature vector of a space–curve is the derivative of the tangent vector $\boldsymbol{t} = d\boldsymbol{x}/ds$ with respect to the arc lenght. The length of $\boldsymbol{\kappa}$ is a measure for the (local) curvature of the space–curve; for example, for a circle with radius R it would be constant $||\boldsymbol{\kappa}|| = 1/R$. The normal curvature is obviously the part of the total curvature which cannot be seen in the tangent plane (since it is perpendicular on it). An ant on the surface of an apple, following a given curve, could not sense the normal curvature as a change in horizontal direction. What an ant could sense, is called the *geodesic* curvature and will be discussed later.

The total curvature vector can be expressed using the local basis:

$$\boldsymbol{t} = \frac{d\boldsymbol{x}}{ds} = \frac{\partial \boldsymbol{x}}{\partial u^1}\frac{du^1}{ds} + \frac{\partial \boldsymbol{x}}{\partial u^2}\frac{du^2}{ds} = \boldsymbol{b}_\alpha u^{\alpha\prime}$$

$$\boldsymbol{\kappa} = \frac{d^2\boldsymbol{x}}{ds^2} = \frac{d}{ds}\left(\boldsymbol{b}_\alpha u^{\alpha\prime}\right) = \boldsymbol{b}_\alpha{}' u^{\alpha\prime} + \boldsymbol{b}_\alpha u^{\alpha\prime\prime}$$

where the $'$ means d/ds. The first equation is the decomposition of the tangent vector for a given surface curve (which obviously lies in the tangent plane) with respect to the local basis $\boldsymbol{b}_1, \boldsymbol{b}_2$. The derivatives $u^{1\prime} = du^1/ds$ and $u^{2\prime} = du^2/ds$ completely determine the direction of the tangent vector, and therefore determine the direction of the surface curve.



Decomposition of the tangent vector w.r.t. the local basis

The last relation can be re–arranged (use $d\boldsymbol{b}_\alpha/ds = (\partial\boldsymbol{b}_\alpha/\partial u^1)u^{1\prime} + (\partial\boldsymbol{b}_\alpha/\partial u^2)u^{2\prime}$) to

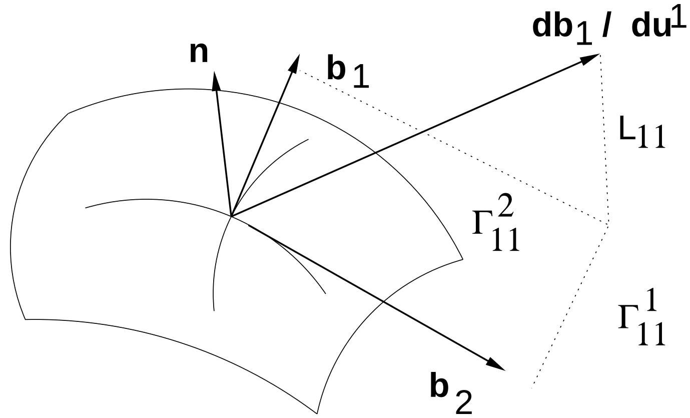
$$\frac{d^2\boldsymbol{x}}{ds^2} = \frac{\partial \boldsymbol{b}_\alpha}{\partial u^\beta} u^{\alpha\prime}u^{\beta\prime} + \boldsymbol{b}_\alpha u^{\alpha\prime\prime}$$

Since $\boldsymbol{b}_\alpha \cdot \boldsymbol{n} = 0$, this means

$$\kappa_n = \frac{\partial \boldsymbol{b}_\alpha}{\partial u^\beta} \cdot \boldsymbol{n}\; u^{\alpha\prime}u^{\beta\prime}$$

It is conveniant to decompose the derivatives of the local basis vectors with respect to the local basis itself,

$$\frac{\partial \boldsymbol{b}_\alpha}{\partial u^\beta} = \Gamma^1_{\alpha\beta}\; \boldsymbol{b}_1 + \Gamma^2_{\alpha\beta}\; \boldsymbol{b}_2 + L_{\alpha\beta}\; \boldsymbol{n} \qquad \alpha, \beta \in \{1, 2\}$$



Derivatives of local basis

$\Gamma^1_{\alpha\beta}, \Gamma^2_{\alpha\beta}$ are the so–called Christoffel symbols, sometimes called connection coefficients. They play a major role when dealing with geodesics, but here they may be considered briefly as linear factors or components with respect to the local basis. Since we cannot assume that the $\partial \boldsymbol{b}_\alpha / \partial u^\beta$ vectors are contained in the tangent plane, it is necessary to consider components $L_{\alpha\beta}$ perpendicular to the tangent plane. Inserting the last expression now into the equation for the normal curvature leaves us with

$$\kappa_n = \left(\Gamma^1_{\alpha\beta}\ \boldsymbol{b}_1 + \Gamma^2_{\alpha\beta}\ \boldsymbol{b}_2 + L_{\alpha\beta}\ \boldsymbol{n}\right) \cdot \boldsymbol{n}\ u^{\alpha\prime} u^{\beta\prime}$$

or, since $\boldsymbol{b}_\alpha \cdot \boldsymbol{n} = 0$

$$\kappa_n = L_{\alpha\beta}\ u^{\alpha\prime} u^{\beta\prime} \tag{1.4}$$

This is the 2nd fundamental form. It relates the normal curvature (the curvature of a normal section) to the direction $u^{\alpha\prime}$ of the surface curve. Thus, if the 2nd fundamental tensor $L_{\alpha\beta}$ is known, the normal curvature can be calculated. The entries of $L_{\alpha\beta}$ are obtained from

$$L_{\alpha\beta} = \frac{\partial \boldsymbol{b}_\alpha}{\partial u^\beta} \cdot \boldsymbol{n}$$

rather straightforward, if the Gaussian representation of the surface is given.

On an *ellipsoid of revolution*, when using λ, ϕ–coordinates, one has

$$\boldsymbol{x} = \begin{pmatrix} N\cos\phi\cos\lambda \\ N\cos\phi\sin\lambda \\ N(1-e^2)\sin\phi \end{pmatrix} \quad \boldsymbol{b}_1 = \frac{\partial \boldsymbol{x}}{\partial \lambda} = \begin{pmatrix} -N\cos\phi\sin\lambda \\ N\cos\phi\cos\lambda \\ 0 \end{pmatrix} \quad \boldsymbol{b}_2 = \frac{\partial \boldsymbol{x}}{\partial \phi} = \begin{pmatrix} -M\sin\phi\cos\lambda \\ M\sin\phi\sin\lambda \\ M\cos\phi \end{pmatrix}$$

with

$$N = \frac{a}{W(\phi)} = \frac{a}{\sqrt{1-e^2\sin^2\phi}} \qquad M = \frac{a(1-e^2)}{W^3(\phi)} = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}$$

and therefore

$$g_{\alpha\beta} = \begin{pmatrix} N^2\cos^2\phi & 0 \\ 0 & M^2 \end{pmatrix}$$

For the 2nd fundamental tensor we obtain

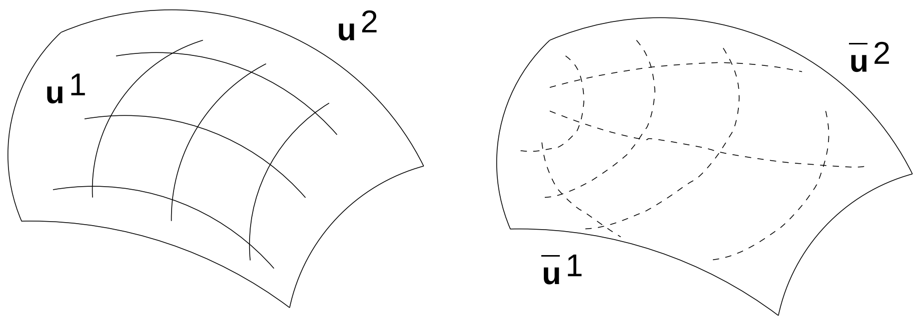
$$L_{\alpha\beta} = \begin{pmatrix} -N\cos^2\phi & 0 \\ 0 & -M \end{pmatrix}$$

1.2 Switching between different coordinate systems

Of course we may use different sets of coordinates or surface parameters in order to describe a surface as a whole or, thinking in terms of geodesy or geo–referencing, to describe positions with respect to this surface. The Gaussian representation of the surface by two different sets of coordinates reads then formally

$$\boldsymbol{x} = \boldsymbol{x}(u^1, u^2) = \bar{\boldsymbol{x}}(\bar{u}^1, \bar{u}^2)$$

They describe the *same* surface, but the functional relations hidden in $\boldsymbol{x}(u^1, u^2)$ and $\bar{\boldsymbol{x}}(\bar{u}^1, \bar{u}^2)$ will be different for different sets of coordinates.



Different coordinates on the same surface

In our applications, one wants to switch between these different coordinates without making use of the (3–dimensional) vector $\boldsymbol{x}$. This is possible, if the *transformation equations* between them are known:

$$u^1 = u^1(\bar{u}^1, \bar{u}^2) \qquad u^2 = u^1(\bar{u}^1, \bar{u}^2) \tag{1.5}$$

$$\bar{u}^1 = \bar{u}^1(u^1, u^2) \qquad \bar{u}^2 = \bar{u}^1(u^1, u^2) \tag{1.6}$$

A trivial example would be 2–dimensional cartesian coordinates $u^\alpha = (x, y)$ and spherical coordinates $\bar{u}^\alpha = (r, \phi)$ on a plane sheet of paper. The transformation equations (1.5) then read explicitly $u^1 = \bar{u}^1 \cdot \cos \bar{u}^2$ and $u^2 = \bar{u}^1 \cdot \sin \bar{u}^2$, and the inverse ones (1.6) read $\bar{u}^1 = \sqrt{(u^1)^2 + (u^2)^2}$ and $\bar{u}^2 = \arctan(u^2/u^1)$.

The situation is quite straightforward, if the definition of a new $\bar{u}^\alpha$ coordinate system starts with the transformation equation. This is often the case in textbooks on differential geometry. In geodesy, unfortunately, the problem is more involved: When defining coordinate systems on ellipsoids of revolution, for instance, our point of departure are certain desired *differential properties* of the new system. The coordinate systems presented in chapter 3 are perfect examples. Typically this leads to the situation that we have some knowledge about partial derivatives of the transformation equations, rather than on the transformation equations themselves. A way out is to write the unknown transformation equations as a Taylor series with two variables, and then to determine the series coefficients (i. e. the partial derivatives) afterwards from the desired properties of the new system. In theory the Taylor series have to be expanded until infinity, in practice it is usually sufficient to term of 5th or 6th order. This means that in total one has to determine 40–60 coefficients. Whether this is really difficult, depends on a clever formulation of the underlying desired differential properties.

A 1–dimensional Taylor (power) series expansions of a function $y = f(x)$ reads

$$y = f(x_0) + \left(\frac{df}{dx}\right)_0 (x - x_0) + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_0 (x - x_0)^2 + \cdots$$

with the *expansion point* x_0. The inverse, as an approximation to $x = f^{-1}(y) = g(y)$ reads

$$x = g(x_0) + \left(\frac{dg}{dy}\right)_0 (y - y_0) + \frac{1}{2}\left(\frac{d^2 g}{dy^2}\right)_0 (y - y_0)^2 + \cdots$$

Here the expansion point was $y_0 = f^{-1}(x_0) = g(x_0)$. Consequently, a 2–timensional Taylor (power) series expansions reads

$$u^1 = (u^1)_0 + \left(\frac{\partial u^1}{\partial \bar{u}^1}\right)_0 \left(\bar{u}^1 - \bar{u}^1_0\right) + \left(\frac{\partial u^1}{\partial \bar{u}^2}\right)_0 \left(\bar{u}^2 - \bar{u}^2_0\right) + \frac{1}{2}\left(\frac{\partial^2 u^1}{\partial (\bar{u}^1)^2}\right)_0 \left(\bar{u}^1 - \bar{u}^1_0\right)^2 + \cdots$$

$$u^2 = (u^2)_0 + \left(\frac{\partial u^2}{\partial \bar{u}^1}\right)_0 \left(\bar{u}^1 - \bar{u}^1_0\right) + \left(\frac{\partial u^2}{\partial \bar{u}^2}\right)_0 \left(\bar{u}^2 - \bar{u}^2_0\right) + \cdots$$

and the inverse one

$$\bar{u}^1 = (\bar{u}^1)_0 + \left(\frac{\partial \bar{u}^1}{\partial u^1}\right)_0 \left(u^1 - u^1_0\right) + \left(\frac{\partial \bar{u}^1}{\partial u^2}\right)_0 \left(u^2 - u^2_0\right) + \cdots$$

$$\bar{u}^2 = (\bar{u}^2)_0 + \left(\frac{\partial \bar{u}^2}{\partial u^1}\right)_0 \left(u^1 - u^1_0\right) + \left(\frac{\partial \bar{u}^2}{\partial u^2}\right)_0 \left(u^2 - u^2_0\right) + \cdots$$

The coordinates of the expansion point, $(u^\alpha)_0, (\bar{u}^\alpha)_0$ have to be known in both systems. Often this is the origin of one of the systems. If the partial derivatives $\partial u^1/\partial \bar{u}^1, \ldots$ are known to sufficient order, the series expansions are as good as a "closed" expression. But what means sufficient? Fortunately the contributions of the higher–order terms will decrease quickly. A common situation in geodetic coordinate systems is, that terms of 6th order and higher give contributions of less than 0.1mm, and can be neglected therefore. This, if course, depends as with all Taylor series on the distance from the expansion point: for larger distances one has to consider more series terms. The statement above concerning geodetic coordinate systems holds for distances of up to 300–500km.

It should be mentioned that, if only the first series (transforming $\bar{u}^\alpha$ into u^α) is known, there are methods ("power series inversion") to find the inverse series in a clever and fast way. This requires the solution of linear equation systems with analytically given coefficients, and is therefore best performed using algebraic manipulation programs like MAPLE or MATHEMATICA. For a deeper insight see Heitz, pp. 67 and 71, and the references he gives.

Once the coefficients are determined (i. e. formulas for the coefficients), the implementation of the coordinate transformation in a computer program is straightforward. In symbolic notation such a program looks like

```
begin: read(ellipsoid_parameters)
       read(coordinates_of_expansion_point)
       compute(series_coefficients)

label: read(coordinates_of_point_in_old_system)
       compute(coordinates_of_point_in_new_system)
       goto label
end:
```

At this point it is probably still unclear what the nature of the above mentioned "desired differential properties" could be? To understand this in the next chapter, we need some preparations: When creating a new coordinate system, say, by applying a certain transformation

$$\bar{u}^\alpha = \bar{u}^\alpha(u^\beta)$$

what happens to the metric tensor? This is clearly an important question, since the metric tensor components determine the local scaling as well as orthogonality, for example. The line element ds itself is independant from a particular coordinate representation, thus

$$ds^2 = \bar{g}_{\alpha\beta}\; d\bar{u}^\alpha\; d\bar{u}^\beta = g_{\gamma\delta}\; du^\gamma\; du^\delta$$

(Remember: greek indices run over $\{1,2\}$. When they show up twice as in the last equation, it does not matter whether to use α,β or γ,δ, because we *sum up* over $\{1,2\}$) Since (use again the chain rule!)

$$du^\gamma = \frac{\partial u^\gamma}{\partial \bar{u}^1} d\bar{u}^1 + \frac{\partial u^\gamma}{\partial \bar{u}^2} d\bar{u}^2 \qquad \gamma \in \{1,2\}$$

we have

$$ds^2 = g_{\gamma\delta}\; \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\; \frac{\partial u^\delta}{\partial \bar{u}^\beta}\; d\bar{u}^\alpha\; d\bar{u}^\beta$$

and, by comparison

$$\bar{g}_{\alpha\beta} \;=\; g_{\gamma\delta}\; \frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\; \frac{\partial u^\delta}{\partial \bar{u}^\beta} \tag{1.7}$$

This rule governs the transformation of the components of the metric tensor, when switching to another coordinate system. It may be written in matrix notation as $\bar{\boldsymbol{G}} = \boldsymbol{U}\boldsymbol{G}\boldsymbol{U}^T$. *Note:* Equation (1.7) involves the derivatives of the *inverse* transformation $u^\alpha = u^\alpha(u^\beta)$.

Example: RD–coordinates. It is convenient to use the series expansions, see e.g. [Strang van Hees, Globale en lokale geodetische systemen, p. 29,30]. With $u^1 = \lambda, u^2 = \phi, \bar{u}^1 = x, \bar{u}^2 = y$:

$$\begin{aligned}
u^1 &= \lambda = \lambda_0 + b_{10}(x - x_0) + b_{11}(x - x_0)(y - y_0) + \cdots \\
u^2 &= \phi = \phi_0 + a_{01}(y - y_0) + a_{20}(x - x_0)^2 + \cdots \\
\bar{u}^1 &= x = x_0 + c_{01}(\lambda - \lambda_0) + c_{11}(\phi - \phi_0)(\lambda - \lambda_0) + \cdots \\
\bar{u}^2 &= y = y_0 + d_{10}(\phi - \phi_0) + d_{20}(\phi - \phi_0)^2 + \cdots
\end{aligned}$$

The coefficients are known, e.g. $c_{01} = 0.36 \cdot C_{01} = 0.36 \cdot 190066.98903$. Therefore the derivatives are also series expansions with known coefficients, for example

$$\frac{\partial u^1}{\partial \bar{u}^1} = \frac{\partial \lambda}{\partial x} = b_{10} + b_{11}(y - y_0) + \cdots$$

$$\frac{\partial u^2}{\partial \bar{u}^1} = \frac{\partial \phi}{\partial x} = 2 \cdot a_{20}(x - x_0) + \cdots$$

We have (since $g_{12} = 0$)

$$\begin{aligned}
\bar{g}_{11} &= g_{11}\left(\frac{\partial u^1}{\partial \bar{u}^1}\right)^2 + g_{22}\left(\frac{\partial u^2}{\partial \bar{u}^1}\right)^2 \\
&= N^2 \cos^2\phi \cdot \left(b_{10} + b_{11}(y - y_0) + \cdots\right) + M^2 \cdot \left(2 \cdot a_{20}(x - x_0) + \cdots\right)
\end{aligned}$$

In the next chapters we will learn that $\bar{g}_{22} = \bar{g}_{11}$ for this case.

Questions

1. A surface is given by the explicit form $z = f(x, y)$. Give the Gaussian representation with $u^1 = x$, $u^2 = y$. Give the line element as a function of dx and dy.

2. Why does it hold

$$||\boldsymbol{t}|| = 1$$

for arbitrary surface curves on arbitrary surfaces?

3. How would you compute the length of a meridian of an ellipsoid of revolution, from the equator to the pole?

4. What does it mean if

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & g_{22} \end{pmatrix} \qquad g_{22} = g_{22}(u^1, u^2)$$

holds for a curvilinear coordinate system?

5. The vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ are contained in the tangent plane of a curved surface. Express the angle ψ between $\boldsymbol{a}$ and $\boldsymbol{c}$ by the local components and the metric tensor.

6. The 2nd fundamental tensor can be derived from

$$L_{\alpha\beta} = \frac{\partial \boldsymbol{b}_\alpha}{\partial u^\beta} \cdot \boldsymbol{n}$$

for a given surface. Why can we use alternatively the following formula?

$$L_{\alpha\beta} = -\boldsymbol{b}_\alpha \cdot \frac{\partial \boldsymbol{n}}{\partial u^\beta}$$

Chapter 2

Isothermal Coordinate Systems

2.1 Basics

In general, *isothermal coordinates* (also called isometric or conformal) $\bar{u}^\alpha = (\bar{u}^1, \bar{u}^2)$ are defined by their special *metric tensor*

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} \bar{G}(\bar{u}^\alpha) & 0 \\ 0 & \bar{G}(\bar{u}^\alpha) \end{pmatrix} = \bar{G}(\bar{u}^\alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the scale factor $\bar{G}(\bar{u}^\alpha)$ varies with the position on the surface. Practically this means that we simply claim this property, and lateron ask for the transformation equations. Remember: $\bar{G} \equiv 1$ all over the surface is impossible even for the sphere. (Why?) The line element therefore is given by

$$\boxed{ds^2 = \bar{g}_{\alpha\beta}\; d\bar{u}^\alpha\; d\bar{u}^\beta = \bar{G}(\bar{u}^\alpha)\left((d\bar{u}^1)^2 + (d\bar{u}^2)^2\right)} \tag{2.1}$$

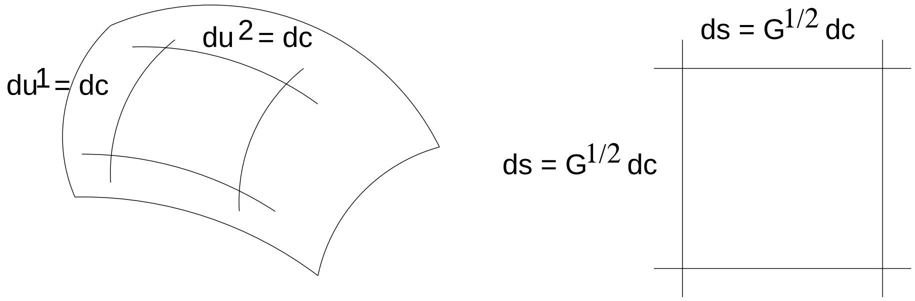
For areas with (at least approximately) *constant* scale factor $\bar{G}$ isothermal coordinates "behave like cartesian coordinates": $\Delta s^2 \approx \bar{G} \cdot (\Delta x^2 + \Delta y^2)$. In geodesy and mapping theory, one always tries to define isothermal coordinate system where the scale factor is close to 1 for a whole area (usually along a meridian) or even for a whole country:

$$\bar{G} = 1 + \delta\bar{G}\,,$$

and consequently

$$\Delta s^2 \approx \Delta x^2 + \Delta y^2 + \delta\bar{G} \cdot \Delta s^2 + \cdots$$

For Δs^2 on the right–hand side, some crude approximation will be sufficient, if $\delta\bar{G}$ is small. From the mapping point of view, (infinitesimal) small *square* grid meshes are mapped onto *square* meshes. Thus, angles remain unchanged, and this kind of mapping is called "conformal".



Infinitesimal grid mesh in isothermal coordinates

Note: This definition says nothing about the coordinate lines. Coordinate lines of isothermal systems are in general *not* geodesics.

2.2 Differential equations

In this section we will see how the property of "being isothermal" can be formulated in terms of partial derivatives; thus, suitable for the construction of transformation equations. For an arbitrary coordinate system $u^\alpha = (u^1, u^2)$ the 1st fundamental form reads

$$ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta$$

By comparison with the desired isothermal coordinates we obtain

$$\bar{g}_{\alpha\beta} \;=\; g_{\gamma\delta}\,\frac{\partial u^\gamma}{\partial \bar{u}^\alpha}\,\frac{\partial u^\delta}{\partial \bar{u}^\beta} = \bar{G}(\bar{u}^\alpha)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We assume now that the original u^α–system is orthogonal, thus $g_{12} = g_{21} = 0$. Then we obtain explicitly

$$\begin{aligned}
\bar{g}_{11} = \bar{G} &= g_{11}\left(\frac{\partial u^1}{\partial \bar{u}^1}\right)^2 + g_{22}\left(\frac{\partial u^2}{\partial \bar{u}^1}\right)^2 \\
\bar{g}_{22} = \bar{G} &= g_{11}\left(\frac{\partial u^1}{\partial \bar{u}^2}\right)^2 + g_{22}\left(\frac{\partial u^2}{\partial \bar{u}^2}\right)^2 \\
\bar{g}_{12} = 0 &= g_{11}\left(\frac{\partial u^1}{\partial \bar{u}^1}\right)\left(\frac{\partial u^1}{\partial \bar{u}^2}\right) + g_{22}\left(\frac{\partial u^2}{\partial \bar{u}^1}\right)\left(\frac{\partial u^2}{\partial \bar{u}^2}\right)
\end{aligned}$$

These expressions are satisfied, if

$$\boxed{\frac{\partial u^1}{\partial \bar{u}^1} = \sqrt{\frac{g_{22}}{g_{11}}}\,\frac{\partial u^2}{\partial \bar{u}^2} \qquad \frac{\partial u^1}{\partial \bar{u}^2} = -\sqrt{\frac{g_{22}}{g_{11}}}\,\frac{\partial u^2}{\partial \bar{u}^1}} \tag{2.2}$$

Or, the other way round,

$$\boxed{\frac{\partial \bar{u}^1}{\partial u^1} = \sqrt{\frac{g_{11}}{g_{22}}}\,\frac{\partial \bar{u}^2}{\partial u^2} \qquad \frac{\partial \bar{u}^1}{\partial u^2} = -\sqrt{\frac{g_{22}}{g_{11}}}\,\frac{\partial \bar{u}^2}{\partial u^1}} \tag{2.3}$$

These are the so–called *differential equations of isothermal coordinates* or conformal mapping. They involve partial derivatives, so they are PDEs. We have 2 relations between the 4 first derivatives, thus only two of the first derivatives will be independent.

Differential equations, which involve *second* and higher partial derivatives, are obtained by successive differentiation of the above given equations. To show how this works, we give an example: From the first relation in (2.2), and by further differentiation with respect to $\bar{u}^1$, it is clear that

$$\frac{\partial^2 u^1}{\partial(\bar{u}^1)^2} = \left[\frac{\partial}{\partial \bar{u}^1}\sqrt{\frac{g_{22}}{g_{11}}}\right]\frac{\partial u^2}{\partial \bar{u}^2} + \sqrt{\frac{g_{22}}{g_{11}}}\left(\frac{\partial^2 u^2}{\partial \bar{u}^2 \partial \bar{u}^1}\right)$$

Probably we will have $g_{\alpha\beta}$ as a function of the u^1, u^2–coordinates instead of $\bar{u}^1, \bar{u}^2$, so it is wise to re–formulate the second derivative using the chain rule

$$\frac{\partial^2 u^1}{\partial(\bar{u}^1)^2} = \left[\left(\frac{\partial}{\partial u^1}\sqrt{\frac{g_{22}}{g_{11}}}\right)\frac{\partial u^1}{\partial \bar{u}^1} + \left(\frac{\partial}{\partial u^2}\sqrt{\frac{g_{22}}{g_{11}}}\right)\frac{\partial u^2}{\partial \bar{u}^1}\right]\frac{\partial u^2}{\partial \bar{u}^2} + \sqrt{\frac{g_{22}}{g_{11}}}\frac{\partial^2 u^2}{\partial \bar{u}^2 \partial \bar{u}^1}$$

The first term in brackets may be replaced using the original PDEs,

$$\frac{\partial^2 u^1}{\partial(\bar{u}^1)^2} = \left[\left(\frac{\partial}{\partial u^1}\sqrt{\frac{g_{22}}{g_{11}}}\right)\sqrt{\frac{g_{22}}{g_{11}}}\frac{\partial u^2}{\partial \bar{u}^2} + \left(\frac{\partial}{\partial u^2}\sqrt{\frac{g_{22}}{g_{11}}}\right)\frac{\partial u^2}{\partial \bar{u}^1}\right]\frac{\partial u^2}{\partial \bar{u}^2} + \sqrt{\frac{g_{22}}{g_{11}}}\frac{\partial^2 u^2}{\partial \bar{u}^2 \partial \bar{u}^1}$$

This is only one of the several relations between the second derivatives (left–hand side and right–hand side). It involves also first derivatives, as well as the metric tensor and its first derivatives.

If the u^α–system is also an isothermal system ($g_{11} = g_{22} = G$), the differential equations above simplify to the *Cauchy–Riemann* differential equations.

$$\boxed{\frac{\partial u^1}{\partial \bar{u}^1} = \frac{\partial u^2}{\partial \bar{u}^2} \qquad \frac{\partial u^1}{\partial \bar{u}^2} = -\frac{\partial u^2}{\partial \bar{u}^1}} \tag{2.4}$$

$$\boxed{\frac{\partial \bar{u}^1}{\partial u^1} = \frac{\partial \bar{u}^2}{\partial u^2} \qquad \frac{\partial \bar{u}^1}{\partial u^2} = -\frac{\partial \bar{u}^2}{\partial u^1}} \tag{2.5}$$

Example: Geodetics ellipsoidal coordinates $u^\alpha = (\lambda, \phi)$ and arbitrary conformal coordinates $\bar{u}^\alpha = (x, y)$. We obtain

$$\frac{\partial x}{\partial \lambda} = \frac{W^2 \cos\phi}{1-e^2}\frac{\partial y}{\partial \phi} \qquad \frac{\partial y}{\partial \lambda} = -\frac{W^2 \cos\phi}{1-e^2}\frac{\partial x}{\partial \phi}$$

This means, in a series expansion

$$x = x_0 + \left(\frac{\partial x}{\partial \lambda}\right)_0 (\lambda - \lambda_0) + \left(\frac{\partial x}{\partial \phi}\right)_0 (\phi - \phi_0) + \cdots$$

$$y = y_0 + \left(\frac{\partial y}{\partial \lambda}\right)_0 (\lambda - \lambda_0) + \left(\frac{\partial y}{\partial \phi}\right)_0 (\phi - \phi_0) + \cdots$$

only 2 of the 4 first order coefficient are independent. Two additional constraints on the first order coefficients will come from the special definition (e.g. transverse Mercator or stereographic) of the system. This can be generally stated by defining a surface curve (the *abscissa line*) explicitly in both coordinate systems, as we will see later.

The fact that the differential equations do not allow a unique determination of the transformation equations, may be seen from a different viewpoint: At the moment, we have agreed about the properties of the coordinates (to be isothermal), but the location of the origin as well as the orientation of the system is still unspecified. Also the "overall scaling", say, a mean value for $\bar{G}$, is still undetermined.

2.3 Complex mapping

In this chapter we consider transformations between *two* isothermal systems, where the Cauchy–Riemann PDEs hold. For this particular case there exists a tricky and elegant representation using complex analysis, which is almost exclusively used in the literature. However, it must be emphasized that in principle everything can be done *without* the complex formulation.

Having said this it makes sense to define a complex variable

$$u = u^1 + iu^2$$

where i is the imaginary unit. A complex number u therefore represents a point on the surface with coordinates u^α. Note: The complex number plane has nothing in common with the tangent plane at the surface! A coordinate transformation can then be described by the complex mapping

$$\boxed{\bar{u} = \bar{u}^1(u^\alpha) + i\,\bar{u}^2(u^\alpha) = f(u)} \tag{2.6}$$

Here f is an arbitrary complex–valued function, and both real and imaginary part of $\bar{u}$ are generally functions of u^α.

It is interesting to consider the partial derivatives of the complex–valued function

$$\frac{\partial \bar{u}}{\partial u^1} = \frac{df}{du}\,\frac{\partial u}{\partial u^1} = \frac{df}{du}$$

$$\frac{\partial \bar{u}}{\partial u^2} = \frac{df}{du}\,\frac{\partial u}{\partial u^2} = i\,\frac{df}{du}$$

Here the chain rule was used. On the other hand, clearly one has

$$\frac{\partial \bar{u}}{\partial u^1} = \frac{\partial \bar{u}^1}{\partial u^1} + i\,\frac{\partial \bar{u}^2}{\partial u^1}$$

$$\frac{\partial \bar{u}}{\partial u^2} = \frac{\partial \bar{u}^1}{\partial u^2} + i\,\frac{\partial \bar{u}^2}{\partial u^2}$$

From the first two relation we have

$$\frac{\partial \bar{u}}{\partial u^1} = \frac{df}{du} = -i\frac{\partial \bar{u}}{\partial u^2}$$

and applying this to the other ones:

$$\frac{\partial \bar{u}^1}{\partial u^1} + i\,\frac{\partial \bar{u}^2}{\partial u^1} = -i\,\frac{\partial \bar{u}^1}{\partial u^2} + \frac{\partial \bar{u}^2}{\partial u^2}$$

Separation into real and imaginary part finally yields

$$\frac{\partial \bar{u}^1}{\partial u^1} = \frac{\partial \bar{u}^2}{\partial u^2} \qquad \frac{\partial \bar{u}^1}{\partial u^2} = -\frac{\partial \bar{u}^2}{\partial u^1}$$

This means that the real and imaginary part of an arbitrary complex–valued transformation *automatically* satisfy the Cauchy–Riemann PDEs, and therefore generate a new isothermal system of coordinates.

Especially useful in geodesy and mapping theory are complex power series

$$\bar{u} = f(u) = \sum_{n=0}^{N} \bar{a}_n (u - u_0)^n \tag{2.7}$$

with complex coefficients $\bar{a}_n = \bar{a}^1_n + i\ \bar{a}^2_n$.

We show for a 1st–order expansion ($N = 1$), that the Cauchy–Riemann PDEs in fact are satisfied:

$$\begin{aligned}\Delta\bar{u} = \bar{u} - \bar{a}_0 = \bar{a}_1(u - u_0) &= \bar{a}_1 \Delta u \\ &= (\bar{a}^1_1 + i\ \bar{a}^2_1)(\Delta u^1 + i\Delta u^2) \\ &= \bar{a}^1_1 \Delta u^1 + i\ \bar{a}^2_1 \Delta u^1 + i\ \bar{a}^1_1 \Delta u^2 - \bar{a}^2_1 \Delta u^2 \\ &\overset{!}{=} \Delta\bar{u}^1 + i\ \Delta\bar{u}^2\end{aligned}$$

Real– and imaginary part are

$$\begin{aligned}\mathrm{Re}(\Delta\bar{u}) &= \Delta\bar{u}^1 = \bar{a}^1_1 \Delta u^1 - \bar{a}^2_1 \Delta u^2 \\ \mathrm{Im}(\Delta\bar{u}) &= \Delta\bar{u}^2 = \bar{a}^2_1 \Delta u^1 + \bar{a}^1_1 \Delta u^2\end{aligned}$$

Thus, in fact

$$\frac{\partial \bar{u}^1}{\partial u^1} = \frac{\partial \Delta\bar{u}^1}{\partial \Delta u^1} = \bar{a}^1_1 = \frac{\partial \Delta\bar{u}^2}{\partial \Delta u^2} = \frac{\partial \bar{u}^2}{\partial u^2}$$

$$\frac{\partial \bar{u}^1}{\partial u^2} = \frac{\partial \Delta\bar{u}^1}{\partial \Delta u^2} = -\bar{a}^2_1 = -\frac{\partial \Delta\bar{u}^2}{\partial \Delta u^1} = -\frac{\partial \bar{u}^2}{\partial u^1}$$

2.4 Role of the abscissa line

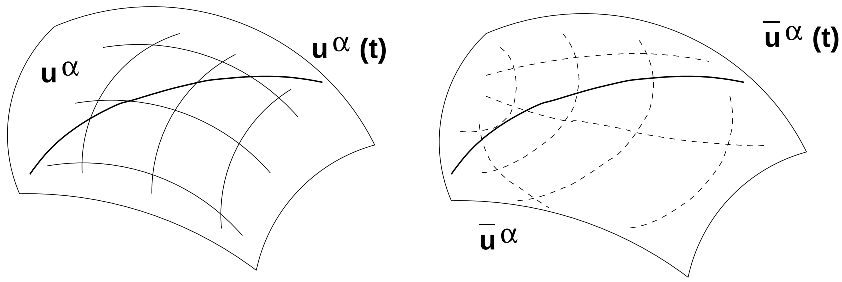
We will see in this chapter how an isothermal coordinate system can be defined in a *unique* way. This means, we will fix the origin and orientation of the system, and find the remaining constraints for the determination of the power series coefficients.

The basic idea is to agree on a certain surface curve (the *abscissa line*) in the "old" u^α–system, preferably represented by

$$u^\alpha = u^\alpha(t) \tag{2.8}$$

where t is a curve parameter, and to attach coordinates of the new system to this curve:

$$\bar{u}^\alpha = \bar{u}^\alpha(t) \tag{2.9}$$



Abscissa line in different coordinate systems

"Attach" means that we simply claim the coordinates (2.9) for the curve. This is a very general description, allowing for arbitrarily strange coordinates. In practice, one would of course choose a "simple" curve. An *example*, which will show up later again, would be to choose a certain meridian with latitude λ_0 on the ellipsoid of revolution as abscissa line. In this case is it quite natural to use the geodetic latitude as a curve parameter $t = \phi$:

$$\left(u^1(\phi), u^2(\phi)\right) = \left(\lambda(\phi), \phi(\phi)\right) = (\lambda_0, \phi)$$

For the isothermal coordinates along the meridian one is tempted to choose

$$\left(\bar{u}^1(\phi), \bar{u}^2(\phi)\right) = \left(S(\phi - \phi_0), 0\right)$$

where $S(\phi - \phi_0)$ would be the distance along the meridian arc, measured from a point with latitude ϕ_0. Obviously the point with geodetic coordinates λ_0, ϕ_0 would be the origin of the isothermal $\bar{u}^\alpha$ system, and for all points along the meridian one would have $\bar{u}^2 = 0$.

What can be said about the partial derivatives? If the representation of the curve is known in both systems as a function of t, it makes sense to consider the derivatives with respect to t:

$$\frac{du^1}{dt} = \frac{\partial u^1}{\partial \bar{u}^1}\frac{d\bar{u}^1}{dt} + \frac{\partial u^1}{\partial \bar{u}^2}\frac{d\bar{u}^2}{dt} \tag{2.10}$$

The left–hand side (du^1/dt) is knowns, since $u^1(t)$ is known. On the right hand side $d\bar{u}^1/dt$ and $d\bar{u}^2/dt$ are also known, since $\bar{u}^1(t)$ and $\bar{u}^2(t)$ are known. This means we have something of the type

$$c = ax + by$$

thus a linear equation with known coefficients a, b, c and two unknowns x, y. The second linear equation comes from

$$\frac{du^2}{dt} = \frac{\partial u^2}{\partial \bar{u}^1}\frac{d\bar{u}^1}{dt} + \frac{\partial u^2}{\partial \bar{u}^2}\frac{d\bar{u}^2}{dt}$$

But this introduces two new unknown partial derivatives, so we have now two linear equations with given coefficients (du^α/dt and $d\bar{u}^\alpha/dt$).

Together with the differential equations of isothermal coordinates, we have four linear equations for the four unknown partial derivatives of the first order. So the series coefficients can in fact be computed, by solving a four–by–four linear equation system!

Linear equations for the higher partial derivatives can be found by successive differentiation of the abscissa line representation, for example:

$$\begin{aligned}
\frac{d^2(u^1)}{dt^2} &= \frac{d}{dt}\left(\frac{\partial u^1}{\partial \bar{u}^1}\frac{d\bar{u}^1}{dt} + \frac{\partial u^1}{\partial \bar{u}^2}\frac{d\bar{u}^2}{dt}\right) \\
&= \frac{d}{dt}\left(\frac{\partial u^1}{\partial \bar{u}^1}\right)\frac{d\bar{u}^1}{dt} + \frac{\partial u^1}{\partial \bar{u}^1}\frac{d^2(\bar{u}^1)}{dt^2} + \frac{d}{dt}\left(\frac{\partial u^1}{\partial \bar{u}^2}\right)\frac{d\bar{u}^2}{dt} + \frac{\partial u^1}{\partial \bar{u}^2}\frac{d^2(\bar{u}^2)}{dt^2} \\
&= \left(\frac{\partial^2 u^1}{\partial (\bar{u}^1)^2}\frac{d\bar{u}^1}{dt} + \frac{\partial^2 u^1}{\partial \bar{u}^1 \partial \bar{u}^2}\frac{d\bar{u}^2}{dt}\right)\frac{d\bar{u}^1}{dt} + \frac{\partial u^1}{\partial \bar{u}^1}\frac{d^2(\bar{u}^1)}{dt^2} \\
&+ \left(\frac{\partial^2 u^1}{\partial \bar{u}^2 \bar{u}^1}\frac{d\bar{u}^1}{dt} + \frac{\partial^2 u^1}{\partial (\bar{u}^2)^2}\frac{d\bar{u}^2}{dt}\right)\frac{d\bar{u}^2}{dt} + \frac{\partial u^1}{\partial \bar{u}^2}\frac{d^2(\bar{u}^2)}{dt^2}
\end{aligned}$$

The relations get quickly complicated, but we can can see that

- 1st and 2nd derivatives with respect to t show up, which are known (from the curve representation)

- 1st order partial derivatives show up, which are known (from the previous four–by–four equation system)

- 2nd order partial derivatives show up: in total 6 unknowns, for which we get now two equations (the 2nd one follows from $d^2(u^2)/dt^2$)

Four relations come from the differential equations of isothermal coordinates. Again, this kind of problem is typically best solved by algebraic manipulation problems like Mathematica or Maple.

Questions

1. We have seen that for the 4 1st–order terms of the power series 2 relations come from the differential equations, and 2 relations from the abscissa line.

   Hoe does the picture look like for the 2nd and 3rd order? And what changes if the complex series are used?

2. Show that for a 2nd order expansion of a complex power series ($N = 2$)

$$\Delta\bar{u} = \bar{a}_1(u - u_0) + \bar{a}_2(u - u_0)^2$$

   the Cauchy–Riemann PDEs are fulfilled!

Chapter 3

Isothermal Geodetic Coordinate Systems

3.1 Mercator–Coordinates

Mercator–coordinates are also called *isothermal geodetic coordinates.* For ellipsoidal geodetic coordinates we have the line element

$$ds^2 = N^2 \cos^2 \phi (d\lambda)^2 + M^2 (d\phi)^2$$

They are clearly not isothermal, since $g_{11} \neq g_{22}$. But when defining the *isothermal latitude* q, by the relation

$$\boxed{(dq)^2 := \frac{M^2}{N^2 \cos^2 \phi} \, (d\phi)^2}$$

we obtain a new coordinate system $\bar{u}^\alpha = (\lambda, q)$ with the desired property of conformity

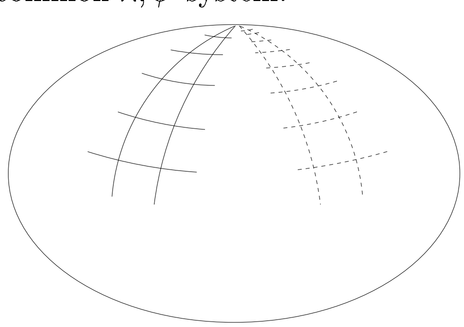
$$ds^2 = N^2 \cos^2 \phi \left((d\lambda)^2 + (dq)^2 \right)$$

These coordinates λ, q are called isothermal geodetic coordinates or Mercator–coordinates. They are, as we have seen, obtained by a simple re–scaling from the geodetic coordinates. The scale factor between the line element and the coordinate differentials is given by $G = N^2 \cos^2 \phi$. However, for the polar regions $\phi \to \pi/2$ obviously one runs into trouble.

Since always $M \geq N$ and $\cos \phi \leq 1$, for the differentials of the latitude must hold

$$dq \geq d\phi$$

This means that the parallels $q = const.$ of the λ, q–system will be "denser" than the parallels $\phi = const.$ of the common λ, ϕ–system:



Geodetic coordinates λ, ϕ (left) and isothermal geodetic coordinates λ, q (right)

Up to now, we have a *differential* re–scaling between the latitudes:

$$dq = \frac{M}{N \cos\phi}\, d\phi = \frac{1-e^2}{W^2 \cos\phi}\, d\phi = \frac{1}{V^2 \cos\phi}\, d\phi$$

This last expression can be integrated analytically

$$\begin{aligned} q &= \int_{\phi=0}^{\phi} \frac{1}{V^2 \cos\phi}\, d\phi' \\ &= \ln\tan(\pi/4 + \phi/2) + e/2 \ln\left(\frac{1 - e\sin\phi}{1 + e\sin\phi}\right) \end{aligned} \tag{3.1}$$

Together with

$$\lambda = \lambda \tag{3.2}$$

this is already the transformation $\bar{u}^\alpha = \bar{u}^\alpha(u^\beta)$. The longitude does not change, and the isothermal latitude depends only on the geodetic latitude. Thus, meridians will be mapped on meridians. The respective formula given by Strang van Hees on page 34 is valid only for the sphere, where $e = 0$.

However, the equation above for the latitudes is not handy and difficult to invert (this means to solve for ϕ). Better suited for practical use are series expansions

$$\begin{aligned} q - q_0 &= \frac{\cos\phi_0}{V^2}(1 + \tan^2\phi_0)\ (\phi - \phi_0) \\ &\quad - \frac{\cos\phi_0}{2V^4}(1 + \tan^2\phi_0)(2 - 3V^2)\ (\phi - \phi_0)^2\ +\ \cdots \end{aligned}$$

$$\phi - \phi_0 = V^2 \cos\phi_0\ (q - q_0)\ +\ \frac{V^2}{2}\cos^2\phi_0 \tan\phi_0 (2 - 3V^2)\ (q - q_0)^2\ +\ \cdots$$

Coefficients up to including 5th order are given, e. g., by Heitz on page 132 and 133. Isothermal geodetic coordinates are extremely useful in geodesy and mapping theory, since the transformation from (λ, q) to another isothermal system can be handled very elegantly with complex power series. They serve as an intermediate coordinate system.

On the sphere, it is common to consider

$$x = R \cdot \lambda \qquad y = R \cdot q$$

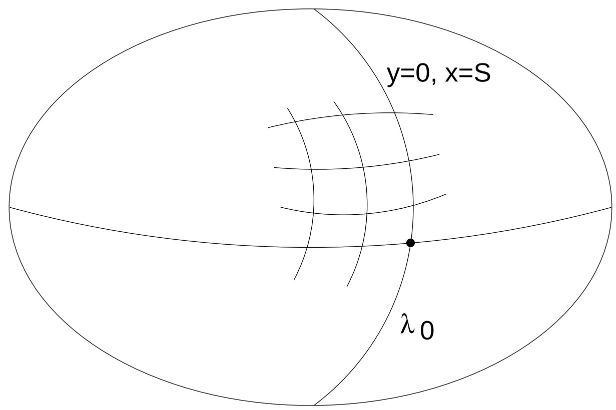
as coordinates. Mercator coordinates on the sphere share the nice property that a "straight line" in Mercator coordinates, this is a curve where

$$x(s) = x_0 + a \cdot (s - s_0) \qquad y(s) = y_0 + b \cdot (s - s_0)$$

holds and which is therefore literally straight in a plane map of (x, y), crosses all meridians under the same angle (azimuth). Such a curve is called a "loxodrome". A loxodrome connecting two points differs from the great circle passing through the same points (the "orthodrome"), but it can be sailed easily: In fact a ship following a constant azimuth (which, at former times, was measured by astronomical and sun observations) sails along a loxodrome. This is the reason why Mercator coordinates played such an important role in the historical development of nautical maps. G. Mercator developed this coordinate system about 1570.

3.2 Transverse Mercator–Coordinates

Transverse Mercator–coordinates are conformal (isothermal) coordinates $\bar{u}^\alpha = (\bar{u}^1, \bar{u}^2) = (x, y)$ on an ellipsoid of revolution. They fulfill the additional condition, that for a certain (central) meridian $\lambda = \lambda_0$ the mapping is equi–distant, and the x–cooordinate equals to the meridional arc length S (distance from equator measured along meridian).



Transverse Mercator–coordinates

This means: For the central meridian, parametrized by the geodetic latitude ϕ in both coordinate systems, holds:

$$\lambda(\phi) = \lambda_0, \qquad \phi = \phi \qquad \text{in geodetic coordinates} \tag{3.3}$$

$$x(\phi) = S(\phi), \qquad y(\phi) = 0 \qquad \text{in transverse Mercator–coordinates} \tag{3.4}$$

In the following, we want to derive transformation equations. A power series expansion of the transformation reads

$$x = x_0 + \left(\frac{\partial x}{\partial \lambda}\right)_0 \Delta\lambda + \left(\frac{\partial x}{\partial \phi}\right)_0 \Delta\phi + \frac{1}{2}\left(\frac{\partial^2 x}{\partial \lambda^2}\right)_0 (\Delta\lambda)^2 + \cdots$$

$$y = y_0 + \left(\frac{\partial y}{\partial \lambda}\right)_0 \Delta\lambda + \left(\frac{\partial y}{\partial \phi}\right)_0 \Delta\phi + \frac{1}{2}\left(\frac{\partial^2 y}{\partial \lambda^2}\right)_0 (\Delta\lambda)^2 + \cdots$$

with

$$\Delta\lambda = \lambda - \lambda_0 \qquad \Delta\phi = \phi - \phi_0$$

The expansion point may be at the equator: $x_0 = 0, y_0 = 0, \phi_0 = 0$, but this is not strictly necessary. λ_0 is the given longitude of the central meridian. Thus the partial derivatives (series coefficients) have to be determined.

For the *1st order*, 4 coefficients have to be determined:

$$\left(\frac{\partial x}{\partial \lambda}\right)_0, \quad \left(\frac{\partial x}{\partial \phi}\right)_0, \quad \left(\frac{\partial y}{\partial \lambda}\right)_0, \quad \left(\frac{\partial y}{\partial \phi}\right)_0$$

The *differential equations* of isothermal coordinates read (see last chapter)

$$\frac{\partial x}{\partial \lambda} = \frac{W^2 \cos\phi}{1-e^2} \frac{\partial y}{\partial \phi} \qquad \frac{\partial y}{\partial \lambda} = -\frac{W^2 \cos\phi}{1-e^2} \frac{\partial x}{\partial \phi}$$

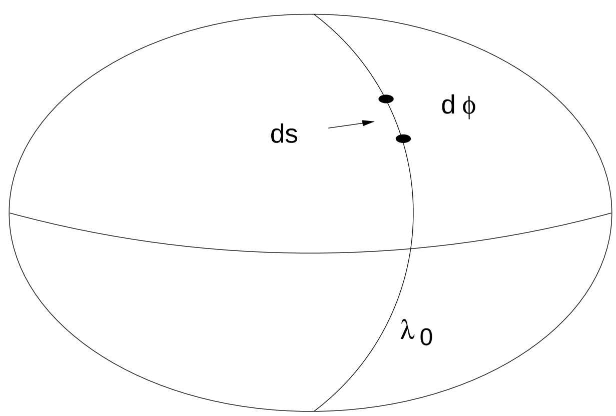
They hold *everywhere* on the surface, and this means, they also hold along the central meridian:

$$\left(\frac{\partial x}{\partial \lambda}\right)_0 = \frac{W_0^2 \cos\phi_0}{1-e^2} \left(\frac{\partial y}{\partial \phi}\right)_0 \qquad \left(\frac{\partial y}{\partial \lambda}\right)_0 = -\frac{W_0^2 \cos\phi_0}{1-e^2} \left(\frac{\partial x}{\partial \phi}\right)_0 \tag{3.5}$$

We have 2 relations. We use (3.4), the representation of the central meridian in the new coordinate system, to obtain additional information:

$$x(\phi) = S(\phi) \quad \Rightarrow \quad \left(\frac{\partial x}{\partial \phi}\right)_0 = \frac{ds}{d\phi} = \sqrt{(g_{22})_0} = M(\phi_0) \tag{3.6}$$

$$y(\phi) = 0 \quad \Rightarrow \quad \left(\frac{\partial y}{\partial \phi}\right)_0 = 0 \tag{3.7}$$



Central meridian and line element

With (3.6), (3.7), and (3.5) the 4 1st derivatives are determined.

2nd order derivatives. In practice, it is necessary to go to higher derivatives. We use, again, the differential equations of isothermal coordinates. They can be differentiated, e.g.

$$\frac{\partial x}{\partial \lambda} = \frac{W^2 \cos\phi}{1-e^2} \frac{\partial y}{\partial \phi} \quad \Rightarrow \quad \frac{\partial^2 x}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{W^2 \cos\phi}{1-e^2}\right) \frac{\partial y}{\partial \phi} + \frac{W^2 \cos\phi}{1-e^2} \frac{\partial^2 y}{\partial \phi \partial \lambda}$$

This leads to

$$\left(\frac{\partial^2 x}{\partial \lambda^2}\right)_0 = \frac{W_0^2 \cos\phi_0}{1-e^2} \left(\frac{\partial^2 y}{\partial\phi\partial\lambda}\right)_0$$

We obtain relations involving the second derivatives. And, again, we can use the representation of the central meridian

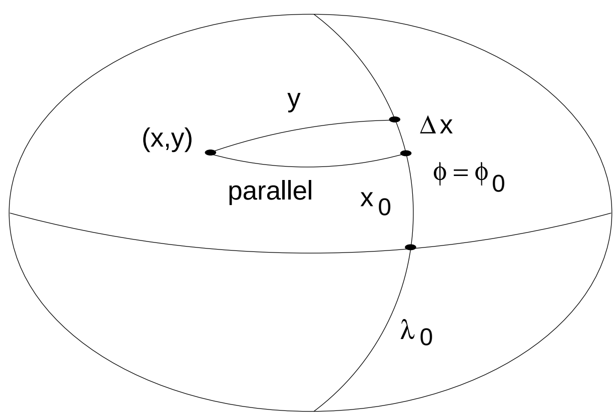
$$\left(\frac{\partial^2 x}{\partial \phi^2}\right)_0 = \frac{d^2 s}{d\phi^2} = \left(\frac{dM}{d\phi}\right)_0$$

$$\left(\frac{\partial^2 y}{\partial \phi^2}\right)_0 = 0$$

In this way, all derivatives (series coefficients) may be computed.

Acceleration. It is common to choose the latitude ϕ_0 of the expansion point to the actual latitude ϕ. This means

- $\Delta\phi = \phi - \phi_0 = 0$ and all parts with $\Delta\phi$ vanish in the series expansion
- The computation is accelerated, since only $\Delta\lambda$–terms remain in the series expansion
- But now we have to know x_0, the arc length from the equator to the parallel with latitude ϕ



Choice of the expansion point

This solution is explicitly given in "Globale en lokale geodetische Systemen", Strang van Hees, p.37.

Transverse Mercator–coordinates are well–suited in the neighbourhood of the central meridian, where the metric factor $\bar{G}_0$ equals to 1. For larger distances from the central meridian, $\bar{G}$ increases. For the *Universal Transverse Mercator* coordinates, the choice $\bar{G}_0 = (0.9996)^2$ was made, in order to keep $\bar{G}$ close to 1 for an extended region. This is achieved by the definition

$$x(\phi) = m \cdot S(\phi)$$

with the factor

$$m = 0.9996$$

3.3 UTM–Coordinates

The Universal Transverse Mercator (UTM–) coordinate system was adopted by the U.S. Army in 1947 for designating 'rectangular' coordinates on large scale military maps. It is defined on the Hayford–ellipsoid. UTM is currently used by the United States and NATO armed forces. With the advent of inexpensive GPS receivers, many other users and institutions are adopting the UTM system for coordinates that are simpler to use than geodetic latitude and longitude.

The UTM system divides the earth into 60 zones each 6^o of longitude wide. This means, in total 60 different central meridians with 6^o spacing are in use, and each zone defines an individual transverse mercator system. UTM zones extend from a latitude of 80^o S to 84^o N. In the polar regions the Universal Polar Stereographic (UPS) coordinate system is used.

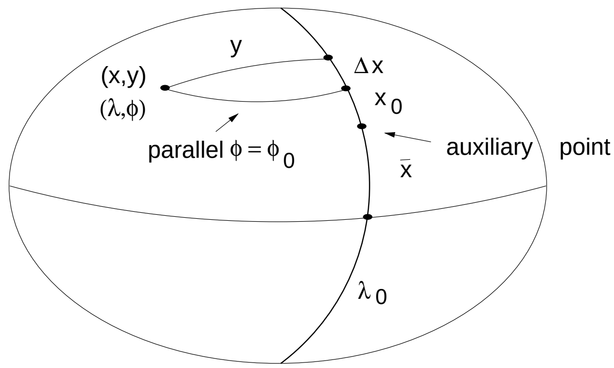
UTM zones are numbered 1 through 60, starting at the international date line, longitude 180^o, and proceeding east. Zone 1 extends from 180^o W to 174^o W, with the central meridian on 177^o W. Important for the Netherlands are zones 31 and 32: Zone 31 covers 0^o E to 6^o E and Zone 32 covers 6^o E to 12^o E. Each zone is divided into horizontal bands spanning 8^o of latitude. These bands are lettered, south to north, beginning at 80^o S with the letter C and ending with the letter X at 84^o N. The letters I and O are skipped to avoid confusion with the numbers one and zero. The band lettered X spans 12^o of latitude.

UTM grid coordinates are expressed as a distance in meters to the east, referred to as the "easting", and a distance in meters to the north, referred to as the "northing". To avoid negative coordinates, 500.000m is generally added to the y coordinate ("false easting"). On the southern hemisphere, 10.000.000m is added to the x coordinate ("false northing").

We give an example for the transformation between geodetic coordinates and UTM–coordinates:

$$\begin{array}{rlll} \text{given coordinates} & \lambda = 5^o.\ 123\ 45 & \phi = 52^o.\ 123\ 45 & \\ \text{to be transformed in zone} & 31 & & \\ \text{parameters Hayford–ellipsoid} & a = 6378388.\ 000 & 1/f = 297.0 & e^2 = 6.722\ 67 \cdot 10^{-3} \end{array}$$

Zone 31 means that we will use the central meridian with longitude $\lambda_0 = 3^o$. Due to numerical reasons (which will become clear in the following) it is wise to introduce an auxiliary point located on the central meridian, for whom both geodetic latitude $\bar{\phi}$ as well as the meridional arc length $\bar{s}$ are already known. Such points are tabulated for the most ellipsoids, and the same point can be used for all computations within the same domain (say, central europe, for example). Moreover, we will use the acceleration technique described in the last chapter, that is, choose the latitude ϕ_0 of the expansion point to the actual latitude ϕ. The picture then looks like



Example: UTM coordinate transformation

Here $\bar{x} = m \cdot \bar{s}$ is the x–coordinate of the auxiliary point, and $x_0 = m \cdot s$ the x–coordinate of the expansion point. The expansion point for the power series is therefore given by

$$\lambda_0 = 9^o \qquad \phi_0 = \phi = 52^o.\ 123\ 45$$

The power series, expanded until the order 5, is then (since $\phi_0 = \phi$ and therefore $\Delta\phi = 0$)

$$\Delta x = \sum_{n=1}^{5} a_n^1 (\Delta\lambda)^n$$
$$\Delta y = y = \sum_{n=1}^{5} a_n^2 (\Delta\lambda)^n$$

with coefficients (see Heitz, page 138, 139, denoted as $\bar{a}_{0n}^{\alpha}$, or Strang van Hees $(p, q, r, \ldots)$), page 37, 38)

$$a_1^1 = 0 \qquad a_1^2 = mN\cos\phi$$
$$a_2^1 = m\frac{N}{2}\cos^2\phi\tan\phi \qquad a_2^2 = 0$$
$$a_3^1 = 0 \qquad a_3^2 = m\frac{N}{6}\cos^3\phi\left(V^2 - \tan^2\phi\right)$$
$$a_4^1 = -m\frac{N}{24}\cos^4\phi\tan\phi\left(4 - 9V^2 + \tan^2\phi\right) \qquad a_4^2 = 0$$
$$a_5^1 = 0 \qquad a_5^2 = m\frac{N}{120}\cos^5\phi\left(5 - 18\tan^2\phi + \tan^4\phi\right)$$

where

$$V = \sqrt{1 + e^2\cos^2\phi} \qquad N = \frac{a}{\sqrt{1 - e^2\sin^2\phi}}$$

Using the values given above, we find

$$\Delta x = 0 + 2\ 126.477 + 0 + 0.309 + 0 = 2\ 126.786$$
$$y = \Delta y = 145\ 381.672 + 0 - 8.158 + 0 - 0.007 = 145\ 373.506$$

Obviously it is not necessary to consider higher–order terms of the expansion. The y–coordinate is already completely determined, but for the x–coordinate

$$x = x_0 + \Delta x = \bar{x} + (x_0 - \bar{x}) + \Delta x$$

the coordinate $x_0 = m \cdot s_0$ of the expansion point still has to be determined. This will be done by numerical integration: Along the meridian we have the relation $ds = \sqrt{g_{22}} d\phi = M(\phi) \cdot d\phi$, which could in principle integrated starting with $\phi = 0$ to $\phi = \phi_0$

$$x_0 = m \cdot s_0 = m \cdot \left(\int_{\phi=0}^{\phi_0} M(\phi) d\phi \right) \approx m \cdot \delta\phi \cdot \left(\sum_{j=1}^{J} M\left((j - \frac{1}{2}) \cdot \delta\phi \right) \right)$$

with J steps and stepsize $\delta\phi = \phi_0 / J$. The problem is that for large distances from the equator (as in our example), the integration is quite difficult to perform: Round–off errors accumulate. The simplest solution is to make use of an auxiliary point with known latitude and meridional arc length (see the figure), for example

$$\bar{\phi} = 48^o.4 \qquad \bar{x} = m \cdot \bar{s} = 5\ 360\ 865.469$$

It remains to solve for $x_0 - \bar{x}$, and this is simply

$$x_0 - \bar{x} = m \cdot (s_0 - \bar{s}) = m \cdot \left(\int_{\phi=\bar{\phi}}^{\phi_0} M(\phi) d\phi \right) \approx m \cdot \delta\phi \cdot \left(\sum_{j=1}^{J} M\left(\bar{\phi} + (j - \frac{1}{2}) \cdot \delta\phi \right) \right)$$

With $J = 100$ steps we find

$$x_0 - \bar{x} = m \cdot 414\ 189.305 = 414\ 023.627$$

To make things shure, we re–do the integration with $J = 1000$ steps and find differences of less than 0.1mm. Therefore the final result for the x–coordinate is

$$x = 2\ 126.786 + 5\ 360\ 865.469 + 414\ 023.627 = 5\ 777\ 015.882$$

and expressed as ‘northing’ and ‘easting’ in zone 31,

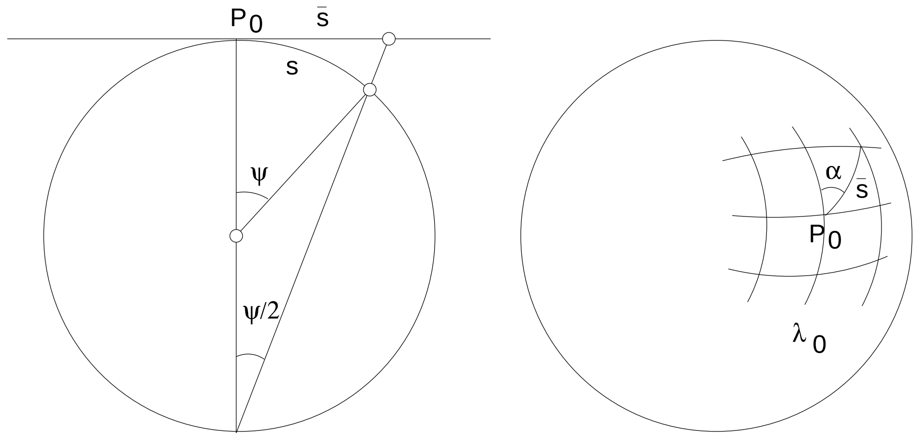
$$N = 5\ 777\ 015.882 \qquad E = 645\ 373.506$$

For comparison, in zone 32 we would find

$$N = 5\ 781\ 979.611 \qquad E = 234\ 642.395$$

3.4 Stereographic Coordinates and RD–Coordinates

Stereographic coordinates are, first of all, considered on the *sphere*. Later we will see how this concept can be used on the ellipsoid of revolution.



Stereographic coordinates on a sphere

A central point P_0 has to be chosen. Stereographic mapping means that the distance s of an arbitrary point on the sphere w. r. t. the central point maps onto the distance

$$\bar{s} = 2R\tan(\psi/2) = 2R\tan(s/(2R))$$

See the figure for a geometric interpretation. Rectangular coordinates $\bar{u}^1 = x, \bar{u}^2 = y$ are then introduced using the spherical azimuth α and the distance $\bar{s}$,

$$x = \bar{s}\cos\alpha + x_0 \qquad y = \bar{s}\sin\alpha + y_0$$

The azimuth refers to the spherical meridian passing through the central point.

Stereographic coordinates on the sphere are isothermal/conformal. An additional property, which is clearly understandable from their definition, is that great circles passing through P_0 are mapped as straight lines. It is one of the reasons why stereographic coordinates are often applied to map the earth's polar regions.

Obviously by differentiation w. r. t. the arc length s

$$d\bar{u}^1/ds = \cos\alpha \; d\bar{s}/ds = \cos\alpha \; \frac{d}{ds}2R\tan(s/(2R)) = \cos\alpha\frac{1}{\cos^2(s/(2R))}$$

$$d\bar{u}^2/ds = \sin\alpha \; d\bar{s}/ds = \sin\alpha \; \frac{d}{ds}2R\tan(s/(2R)) = \sin\alpha\frac{1}{\cos^2(s/(2R))}$$

Since $ds^2 = \bar{g}_{\alpha\beta}d\bar{u}^1 d\bar{u}^2 = \bar{g}_{11}(d\bar{u}^1)^2 + \bar{g}_{22}(d\bar{u}^2)^2$, and for the metric tensor entries hold

$$\bar{G} = \bar{g}_{11} = \bar{g}_{22} = \cos^4(s/(2R)) =: F(s)$$

$\bar{G}$ is a function of the distance s only (this is obvious from the definition of this coordinate system).

For *stereographic coordinates on an ellipsoid of revolution* there is no geometric interpretation. However, they can be introduced in two ways: directly or as a so–called double projection. We follow the the direct (modern) definition.

The idea is as follows: We propose that the mapping properties of spherical stereographic coordinates should at least hold approximately for ellipsoidal stereographic coordinates, where the radius of the sphere should be the Gaussian radius at the central point

$$R_0 = \sqrt{N(\phi_0) \cdot M(\phi_0)} \tag{3.8}$$

We simply *claim* that they hold *exactly* for the meridian $\lambda = \lambda_0$ passing through the central point. That means, for the central meridian (the abscissa line) we have the representation (compare with transverse Mercator–coordinates)

$$\lambda(\phi) = \lambda_0 \qquad \phi = \phi \qquad \text{in geodetic coordinates} \tag{3.9}$$

$$x(\phi) = \bar{u}^1 = 2R_0 \tan(s/(2R_0)) + x_0 \qquad y(\phi) = y_0 \qquad \text{in ell. stereogr. coordinates} \tag{3.10}$$

where $s = s(\phi)$ can be considered as a known function of s

It also means that along the central meridian

$$\bar{G}_0 = \cos^4(s/(2R_0)) = F(s)$$

but for arbitrary points on the ellipsoid only

$$\bar{G} \approx \cos^4(s/(2R_0))$$

Now we can proceed in the same way than with introducing transverse Mercator–coordinates. A power series expansion of the transformation reads

$$x = x_0 + \left(\frac{\partial x}{\partial \lambda}\right)_0 \Delta\lambda + \left(\frac{\partial x}{\partial \phi}\right)_0 \Delta\phi + \frac{1}{2}\left(\frac{\partial^2 x}{\partial \lambda^2}\right)_0 (\Delta\lambda)^2 + \cdots$$

$$y = y_0 + \left(\frac{\partial y}{\partial \lambda}\right)_0 \Delta\lambda + \left(\frac{\partial y}{\partial \phi}\right)_0 \Delta\phi + \frac{1}{2}\left(\frac{\partial^2 y}{\partial \lambda^2}\right)_0 (\Delta\lambda)^2 + \cdots$$

with

$$\Delta\lambda = \lambda - \lambda_0 \qquad \Delta\phi = \phi - \phi_0$$

The partial derivatives (series coefficients) have to be determined, e. g. for the **1st order**

$$\left(\frac{\partial x}{\partial \lambda}\right)_0, \left(\frac{\partial x}{\partial \phi}\right)_0, \left(\frac{\partial y}{\partial \lambda}\right)_0, \left(\frac{\partial y}{\partial \phi}\right)_0$$

From the *differential equations* of isothermal coordinates (see chapter “Isothermal coordinates...”)

$$\left(\frac{\partial x}{\partial \lambda}\right)_0 = \frac{W_0^2 \cos\phi_0}{1-e^2}\left(\frac{\partial y}{\partial \phi}\right)_0 \qquad \left(\frac{\partial y}{\partial \lambda}\right)_0 = -\frac{W_0^2 \cos\phi_0}{1-e^2}\left(\frac{\partial x}{\partial \phi}\right)_0 \tag{3.11}$$

we have 2 relations. We use the representation of the central meridian in the ellipsoidal stereographic coordinate system, to obtain the remaining 2 relations

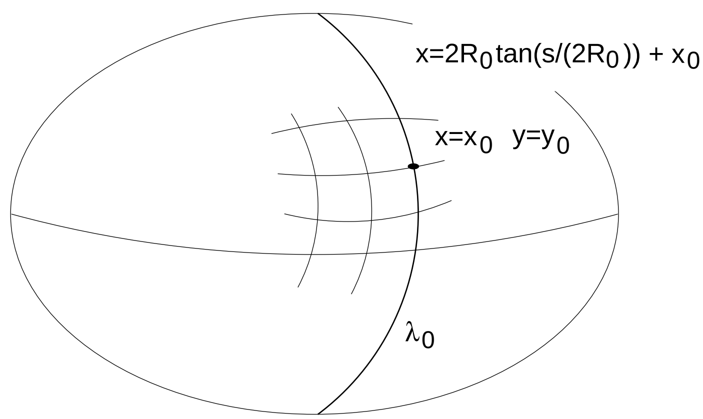
$$x(\phi) = 2R_0 \tan\left(\frac{s(\phi)}{2R_0}\right) + x_0 \qquad \Rightarrow$$

$$\left(\frac{\partial x}{\partial \phi}\right)_0 = 2R_0 \frac{\partial}{\partial \phi}\left(\tan\left(\frac{s(\phi)}{2R_0}\right)\right)_0 = 2R_0 \frac{d}{ds}\left(\tan(s/(2R_0))\right)_0 \left(\frac{ds}{d\phi}\right)_0 = \frac{1}{\cos^2(s/2R_0)} M(\phi_0) \tag{3.12}$$

$$y(\phi) = 0 \qquad \Rightarrow \qquad \left(\frac{\partial y}{\partial \phi}\right)_0 = 0 \tag{3.13}$$

With (3.12), (3.13), and (3.11) the 4 1st–order derivatives are determined. Higher–order derivatives follow in the same way: Always 2 relations from the representation of the central meridian by differentiation, and the remaining relations from differentiation of the differential equations (3.5).

A figure of the coordinate grid looks (qualitatively) similar to the grid of transverse Mercator–coordinates:



Stereographic coordinates on an ellipsoid of revolution

Ellipsoidal stereographic coordinates are in use in the netherlands, and are called *RD–coordinates* (RD = Rijksdriehoeksmeting). The chosen ellipsoid was the Bessel–ellipsoid (which differs from the Hayford–ellipsoid used for UTM–coordinates!). The central point is the station Amersfoort, located at

$$\lambda_0 = 5^o, 387\ 638\ 889 \qquad \phi_0 = 52^o, 156\ 160\ 556$$

This means the central meridian of the system is defined by λ_0 given above. More (and also historical) information can be found in Strang van Hees, chapter 6 and 7). For the (initial) spherical stereographic mapping was chosen

$$\bar{s} = 2\ k\ R \tan(\psi/2) = 2\ k\ R \tan(s/(2R))$$

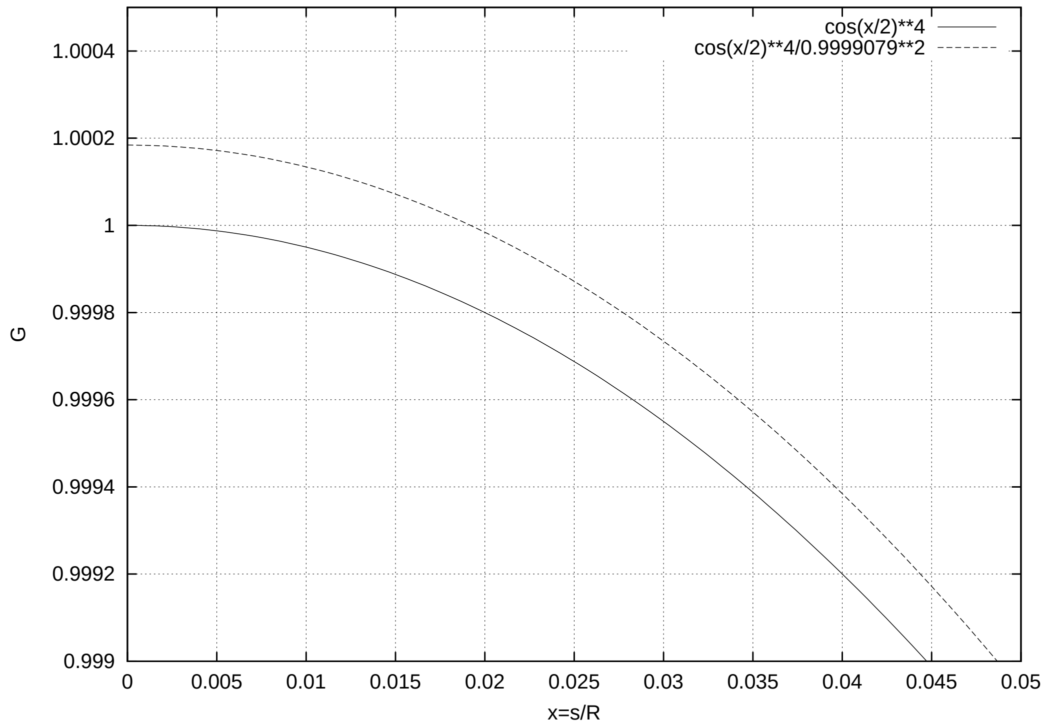
with

$$k = 0.9999079$$

This factor $k < 1$ keeps the metric factor

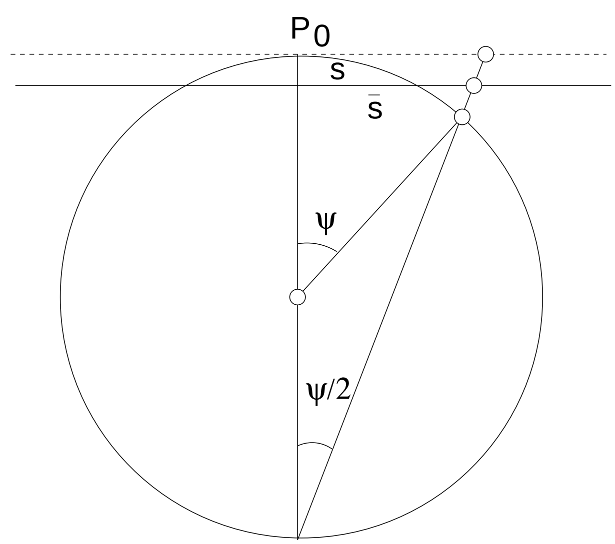
$$\bar{G}_0 = \frac{1}{k^2}\ \cos^4(s/(2R))$$

closer to 1 for a larger area, since $\bar{G}$ decreases for increasing distance s from the central point. The figure shows the functions $\cos^4(s/2R)$ and $\cos^4(s/2R)/k^2$ for $s/R \in [0 \ldots 0.05]$. This means the metric factor $\bar{G}_0$ is approximately 1.0002 at the central point, and approximately 0.999 at 300km distance. Strictly speaking, the figure holds for spherical stereographic coordinates as well as for the central meridian of ellipsoidal stereographic coordinates. The deviations of $\bar{G}$ from $\bar{G}_0$ are much smaller, fortunately, and can hardly be shown in the same diagram.



Metric factor for stereographic coordinates ($k = 1$) and RD–coordinates

For the spherical case this can be seen geometrically as a mapping onto an intersecting plane.



Use of intersecting plane

The series coefficients are given by Strang van Hees, on pages 29,30, or Heitz, page 163, 164.

Questions

1. How would you derive transformation equations for 2 neighbouring UTM–systems (zones) , e. g. $\lambda_1 = -3°$ and $\lambda_2 = +3°$, without going back to geodetic coordinates?

2. RD–coordinates: In many books you will find the (approximate) formula

$$S_{RD} = m \cdot S = k \left(1 + \frac{r^2}{4R^2}\right) S$$

   for the reduction of measured (horizontal) distances S to "RD–distances" $S_{RD} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Here r means the (mean) distance of the end points to the central points. Derive this formula! Hint: Show that $m = (\bar{G}_0)^{-2}$. Can you still work with this formula at 1000km distance from the central point?

3. Why is the factor k for RD–coordinates much closer to one as the factor m for UTM–coordinates?

4. When computing UTM–coordinates in two neighbouring zones for the same station, both x (northing) and y (easting) will differ. Why? Are there points where only y will differ?