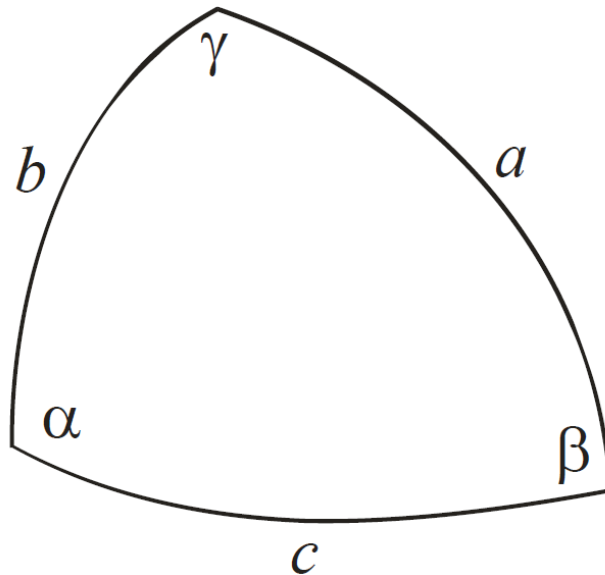


**VAJA 9 – SFERNA TRIGONOMETRIJA**

**1 SPLOŠNI SFERNI TRIKOTNIK**



**Kosinusni izrek za stranice**

$$\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c \cos \alpha \\ \cos b &= \cos a \cos c + \sin a \sin c \cos \beta \\ \cos c &= \cos a \cos b + \sin a \sin b \cos \gamma\end{aligned}$$

**Kosinusni izrek za kote**

$$\begin{aligned}\cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a \\ \cos \beta &= -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b \\ \cos \gamma &= -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c\end{aligned}$$

**Sinusni izrek**

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}$$

## Kotangensni izrek

$$\begin{aligned}\cot a \sin b &= \cos b \cos \gamma + \cot \alpha \sin \gamma \\ \cot b \sin c &= \cos c \cos \alpha + \cot \beta \sin \alpha \\ \cot c \sin b &= \cos b \cos \alpha + \cot \gamma \sin \alpha \\ \cot a \sin c &= \cos c \cos \beta + \cot \alpha \sin \beta \\ \cot b \sin a &= \cos a \cos \gamma + \cot \beta \sin \gamma \\ \cot c \sin a &= \cos a \cos \beta + \cot \gamma \sin \beta\end{aligned}$$

## Napierjeve enačbe (analogije)

$$\tan \frac{\alpha + \beta}{2} = \frac{\cos \frac{a - b}{2}}{\cos \frac{a + b}{2}} \cot \frac{\gamma}{2}$$

$$\tan \frac{\alpha - \beta}{2} = \frac{\sin \frac{a - b}{2}}{\sin \frac{a + b}{2}} \cot \frac{\gamma}{2}$$

$$\tan \frac{\alpha + \gamma}{2} = \frac{\cos \frac{a - c}{2}}{\cos \frac{a + c}{2}} \cot \frac{\beta}{2}$$

$$\tan \frac{\alpha - \gamma}{2} = \frac{\sin \frac{a - c}{2}}{\sin \frac{a + c}{2}} \cot \frac{\beta}{2}$$

$$\tan \frac{\beta + \gamma}{2} = \frac{\cos \frac{b - c}{2}}{\cos \frac{b + c}{2}} \cot \frac{\alpha}{2}$$

$$\tan \frac{\beta - \gamma}{2} = \frac{\sin \frac{b - c}{2}}{\sin \frac{b + c}{2}} \cot \frac{\alpha}{2}$$

$$\tan \frac{a + b}{2} = \frac{\cos \frac{\alpha - \beta}{2}}{\cos \frac{\alpha + \beta}{2}} \tan \frac{c}{2}$$

$$\tan \frac{a - b}{2} = \frac{\sin \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}} \tan \frac{c}{2}$$

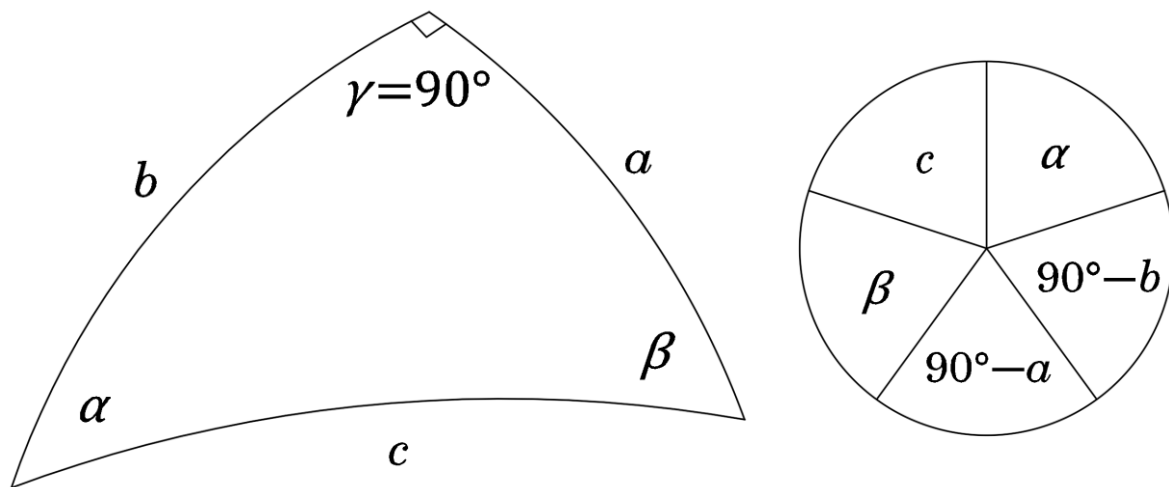
$$\tan \frac{a + c}{2} = \frac{\cos \frac{\alpha - \gamma}{2}}{\cos \frac{\alpha + \gamma}{2}} \tan \frac{b}{2}$$

$$\tan \frac{a - c}{2} = \frac{\sin \frac{\alpha - \gamma}{2}}{\sin \frac{\alpha + \gamma}{2}} \tan \frac{b}{2}$$

$$\tan \frac{b + c}{2} = \frac{\cos \frac{\beta - \gamma}{2}}{\cos \frac{\beta + \gamma}{2}} \tan \frac{a}{2}$$

$$\tan \frac{b - c}{2} = \frac{\sin \frac{\beta - \gamma}{2}}{\sin \frac{\beta + \gamma}{2}} \tan \frac{a}{2}$$

## 2 PRAVOKOTNI SFERNI TRIKOTNIK



### Napierjevo pravilo

Kosinus izbranega elementa v "krogu" (shema desno) je enak (shema zgoraj desno):

- i) produktu kotangensov sosednjih elementov,
- ii) produktu sinusov nasprotnih elementov.

#### Osnovna oblika izraza

$$\cos c = \cot \beta \cot \alpha$$

$$\cos c = \sin(90^\circ - a) \sin(90^\circ - b)$$

$$\cos \alpha = \cot c \cot(90^\circ - b)$$

$$\cos \alpha = \sin \beta \sin(90^\circ - a)$$

$$\cos(90^\circ - b) = \cot \alpha \cot(90^\circ - a)$$

$$\cos(90^\circ - b) = \sin c \sin \beta$$

$$\cos(90^\circ - a) = \cot(90^\circ - b) \cot \beta$$

$$\cos(90^\circ - a) = \sin \alpha \sin c$$

$$\cos \beta = \cot(90^\circ - a) \cot c$$

$$\cos \beta = \sin(90^\circ - b) \sin \alpha$$

#### Poenostavljena oblika izraza

$$\cos c = \cot \beta \cot \alpha$$

$$\cos c = \cos a \cos b$$

$$\cos \alpha = \cot c \tan b$$

$$\cos \alpha = \sin \beta \cos a$$

$$\sin b = \cot \alpha \tan a$$

$$\sin b = \sin c \sin \beta$$

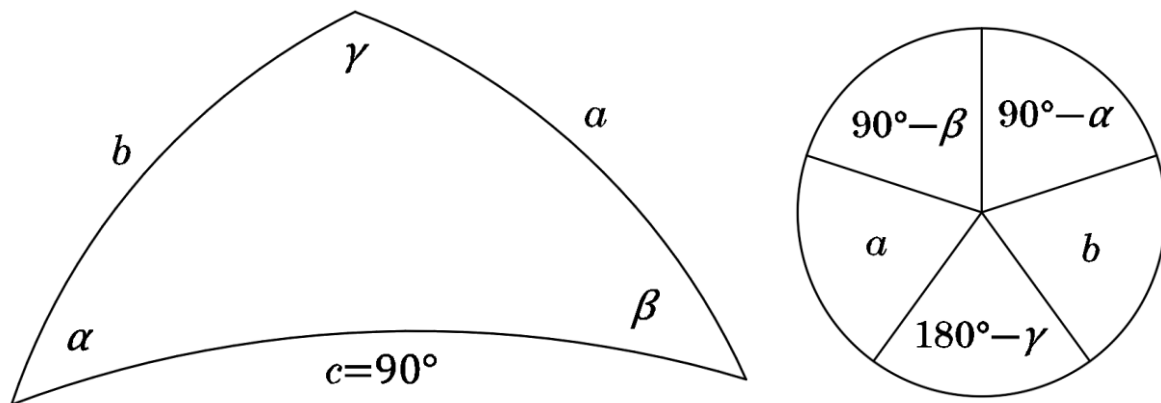
$$\sin a = \tan b \cot \beta$$

$$\sin a = \sin \alpha \sin c$$

$$\cos \beta = \tan a \cot c$$

$$\cos \beta = \cos b \sin \alpha$$

### 3 PRAVOSTRANIČNI SFERNI TRIKOTNIK



#### Napierjevo pravilo

Kosinus izbranega elementa v "krogu" (shema desno) je enak (shema zgoraj desno):

- i) produktu kotangensov sosednjih elementov,
- ii) produktu sinusov nasprotnih elementov.

#### Osnovna oblika izraza

$$\begin{aligned} \cos(90^\circ - \alpha) &= \cot(90^\circ - \beta) \cot b \\ \cos(90^\circ - \alpha) &= \sin a \sin(180^\circ - \gamma) \\ \cos b &= \cot(90^\circ - \alpha) \cot(180^\circ - \gamma) \\ \cos b &= \sin(90^\circ - \beta) \sin a \\ \cos(180^\circ - \gamma) &= \cot b \cot a \\ \cos(180^\circ - \gamma) &= \sin(90^\circ - \alpha) \sin(90^\circ - \beta) \\ \cos a &= \cot(180^\circ - \gamma) \cot(90^\circ - \beta) \\ \cos a &= \sin b \sin(90^\circ - \alpha) \\ \cos(90^\circ - \beta) &= \cot a \cot(90^\circ - \alpha) \\ \cos(90^\circ - \beta) &= \sin(180^\circ - \gamma) \sin b \end{aligned}$$

#### Poenostavljena oblika izraza

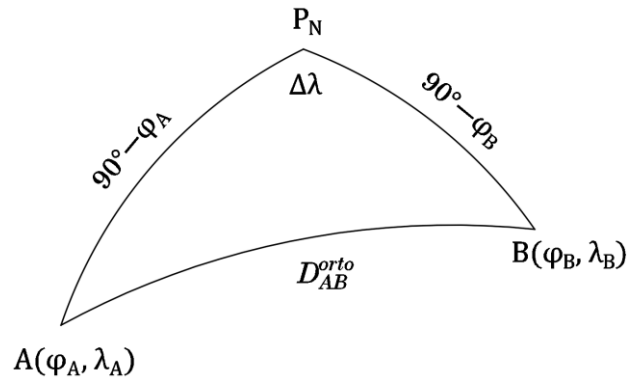
$$\begin{aligned} \sin \alpha &= \tan \beta \cot b \\ \sin \alpha &= \sin a \sin \gamma \\ \cos b &= -\tan \alpha \cot \gamma \\ \cos b &= \cos \beta \sin a \\ \cos \gamma &= -\cot b \cot a \\ \cos \gamma &= -\cos \alpha \cos \beta \\ \cos a &= -\cot \gamma \tan \beta \\ \cos a &= \sin b \cos \alpha \\ \sin \beta &= \cot a \tan \alpha \\ \sin \beta &= \sin \gamma \sin b \end{aligned}$$

## 4 ORTODROMA

Ortodroma oziroma geodetska linija je najkrajša razdalja med dvema točkama na krogli. Ortodroma je krajši lok velikega kroga skozi dani točki na površju krogle. Kot pot plovbe/leta ni primerna saj bi moralo plovilo/letalo za pot po ortodromi stalno spreminjati smer (kurz) poti.

dano:  $A(\varphi_A, \lambda_A), B(\varphi_B, \lambda_B)$

iščemo:  $D_{AB}^{orto}$



Dolžino ortodrome v kotnih enotah izračunamo z uporabo kosinusnega izrek za stranice:

$$\cos D_{AB}^{orto} = \cos(90^\circ - \varphi_A) \cos(90^\circ - \varphi_B) + \sin(90^\circ - \varphi_A) \sin(90^\circ - \varphi_B) \cos \Delta\lambda$$

kjer je:

$$\Delta\lambda = |\lambda_B - \lambda_A|$$

Poldnevniko so veliki krogi → loki poldnevnikov so torej ortodrome. Vzporedniki so mali krogi (razen ekvator) → loki vzporednikov NISO ortodrome (razen loki ekvatorja).

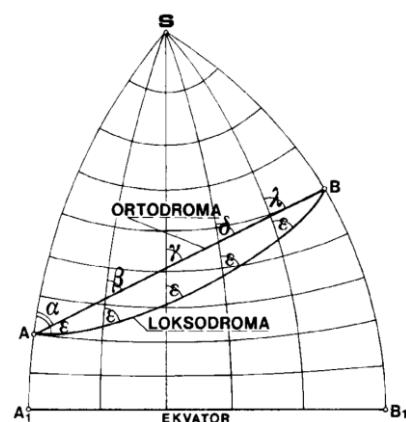
## 5 LOKSODROMA

Loksodroma je pot med dvema točkama, ki vse meridiane seka pod enakim kotom.

Plovilu/letalu, ki potuje po loksodromi, ni potrebno spreminjati kurza poti.

dano: začetna točka  $A(\varphi_A, \lambda_A)$ , končna točka  $B(\varphi_B, \lambda_B)$

iščemo:  $D_{AB}^{loksodromi}, \alpha$



Loksodroma ni lok velikega kroga, zato ne moremo uporabiti enačb sferne trigonometrije. Azimut loksodrome (smer poti oziroma kurz)  $\alpha$  (na zgornji skici označen z  $\varepsilon$ ) izračunamo kot:

$$\cot \alpha = \frac{1}{(\lambda_B - \lambda_A) \frac{\pi}{180^\circ}} \ln \left( \frac{\tan \left( 45^\circ + \frac{\varphi_B}{2} \right)}{\tan \left( 45^\circ + \frac{\varphi_A}{2} \right)} \right)$$

Za končni izračun azimuta je potrebno določiti še njegov kvadrant:

$\lambda_A - \lambda_B$	$\cot \alpha$	kvadrant	$\alpha$
-	+	I. kvadrant	$\alpha$
-	-	II. kvadrant	$\alpha + 180^\circ$
+	+	III. kvadrant	$\alpha + 180^\circ$
+	-	IV. kvadrant	$\alpha + 360^\circ$

Dolžino loksodrome v dolžinskih enotah izračunamo kot:

$$D_{AB}^{loks} = R \frac{(\varphi_B - \varphi_A)}{\cos \alpha} \frac{\pi}{180^\circ}$$